MECHANICS OF FLUIDS

Lecture 5 – Integral Formulation
Reynolds theorem and Mass conservation
Lecturer: Hamidreza Norouzi

Note

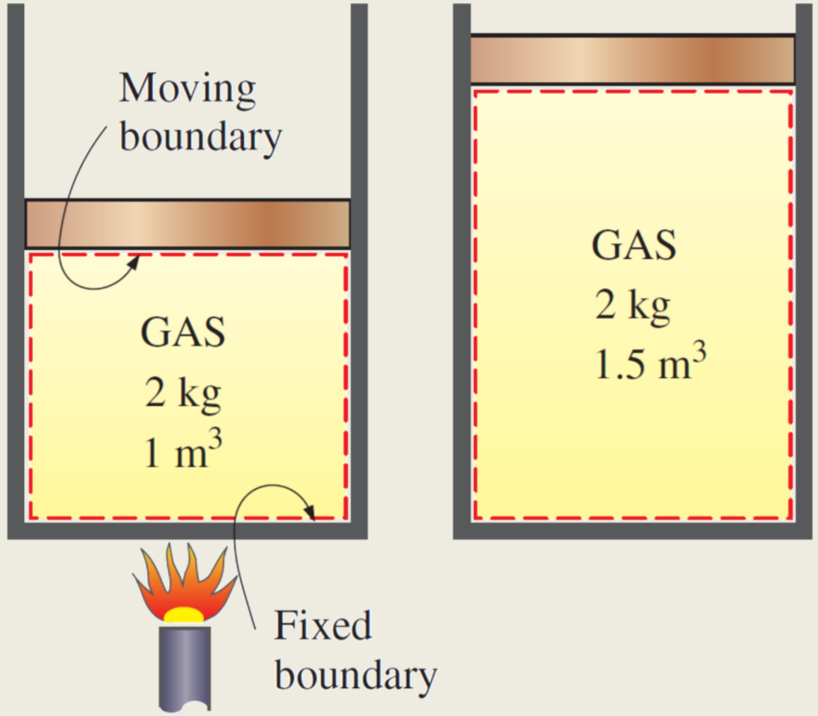
- All the art-work contents of this lecture are obtained from the following sources, unless otherwise stated:
  - *Fluid Mechanics, 8th edition, Frank M. White, McGraw-Hill, 2016.*
  - *Fluid Mechanics: Fundamental and Applications, 3rd edition, Yunus A. Cengel, John M. Cimbala, McGraw-Hill, 2014.*

Basic concepts

- **System:** quantity of matter or a region in space chosen for Study. region outside the system is called the **surroundings.**
- The real or imaginary surface that separates the system from its surroundings is called the **boundary.**

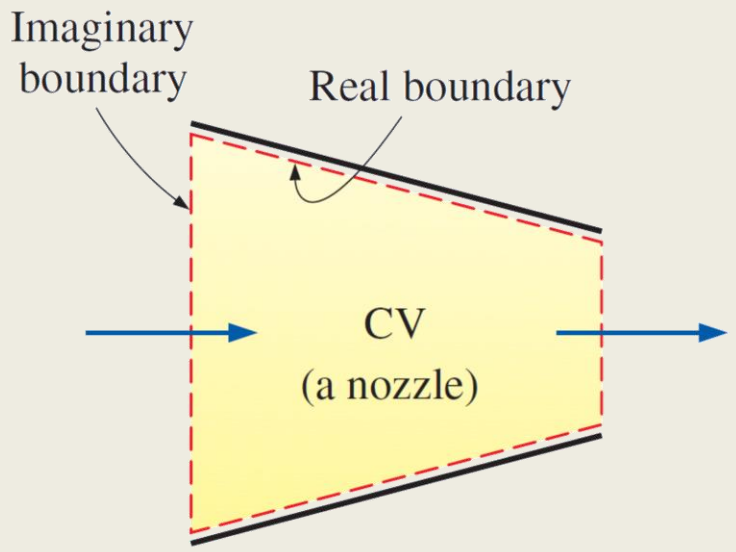
- **Control mass (closed system):**
  - *Mass is fixed and does not cross the boundaries*
  - *Energy in the form of work and heat can cross the boundaries.*
  - *The boundaries can be fixed or moving*

  - *A closed system with no energy exchange is called **isolated system***



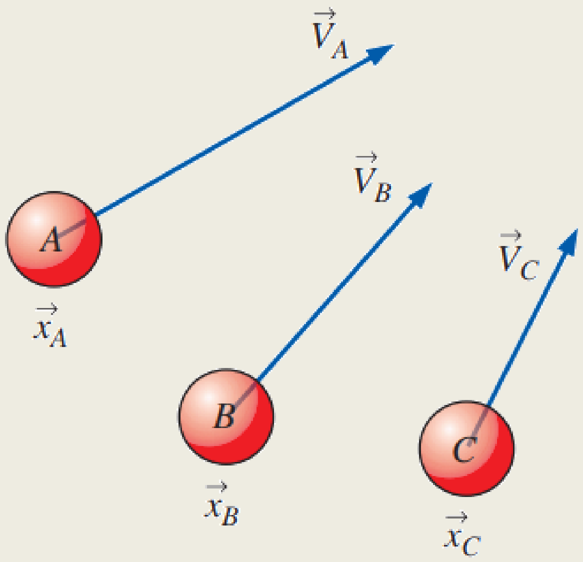
Basic concepts

- **Control volume (open system):**
  - *Both mass and energy can cross the boundaries of the control volume*

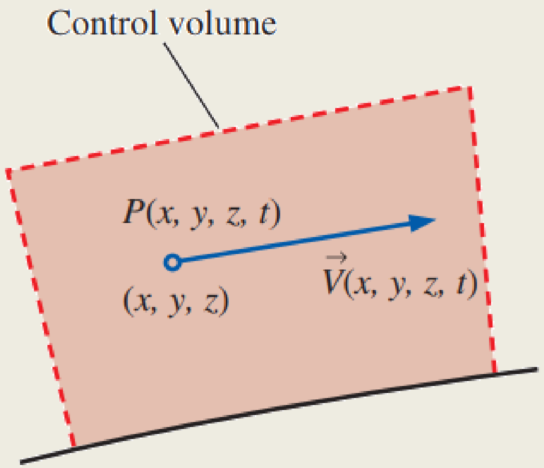


Basic concepts

- Lagrangian formulation



- Eulerian formulation



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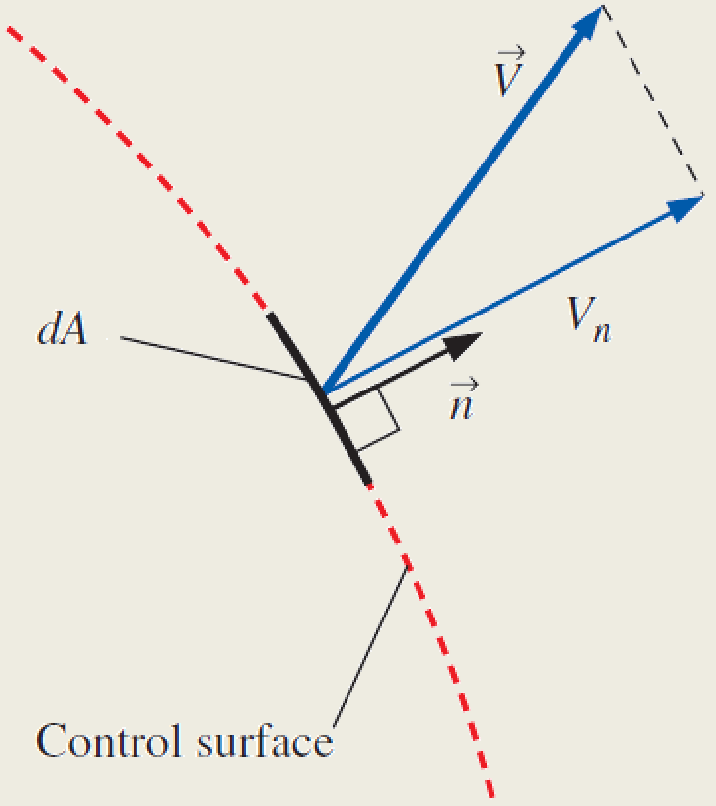
Basic concepts

- Basic laws in mechanics of fluids
  - *First law for a closed system: conservation of mass*
  - *Second law for a closed system: conservation of linear momentum (Newton’s second law)*
  - *Third law for a closed system: conservation of energy (first law of thermodynamic)*

Mass & volume flow across a control surface

- The volume of fluid (*dV*) which passes through and elemental area *dA* during *dt* seconds is:

$$d\mathcal{V} = V\,dt\,dA\cos\theta = (\mathbf{V}\cdot\mathbf{n})\,dA\,dt$$



- The total flow rate of fluid through the surface:

$$Q = \int_s (\mathbf{V}\cdot\mathbf{n})\,dA = \int_s V_n\,dA$$

- and mass flow rate:

$$\dot{m} = \int_s \rho(\mathbf{V}\cdot\mathbf{n})\,dA = \int_s \rho V_n\,dA$$

Mass & volume flow across a control surface

- If density is constant, average velocity is obtained by (volume-average velocity):

$$V_{av} = \frac{Q}{A} = \frac{1}{A}\int (\mathbf{V} \cdot \mathbf{n})\, dA$$

- And if density is not constant, average density is obtained:

$$\rho_{av} = \frac{1}{A}\int \rho\, dA$$

- mass flow rate can be approximated by:

$$\dot{m} = \int_s \rho(\mathbf{V} \cdot \mathbf{n})\, dA \approx V_{av}\, \rho_{av}\, A$$

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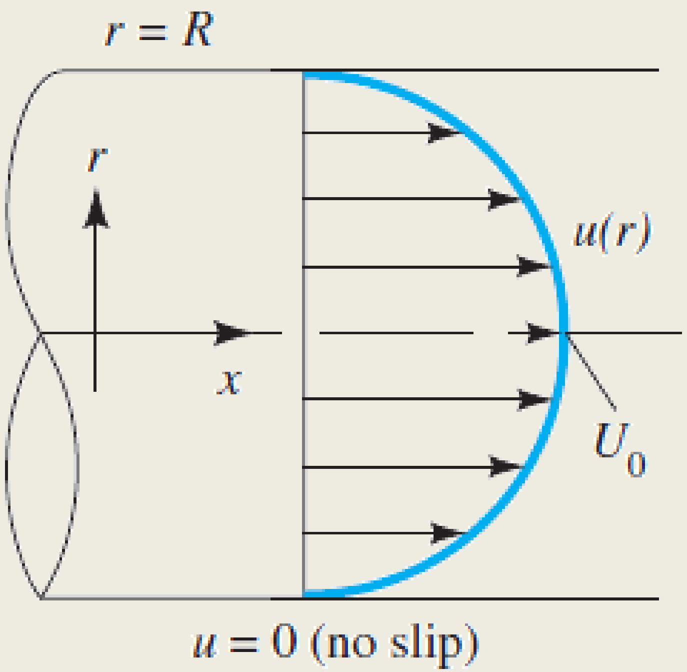
Example

For steady viscous flow through a circular tube, the axial velocity profile is given approximately by:

$$u = U_0 \left(1 - \frac{r}{R}\right)^m$$



For laminar flow m ≈ 1/2 and for turbulent flow m ≈ 1/7. Compute the average velocity if the density is constant.

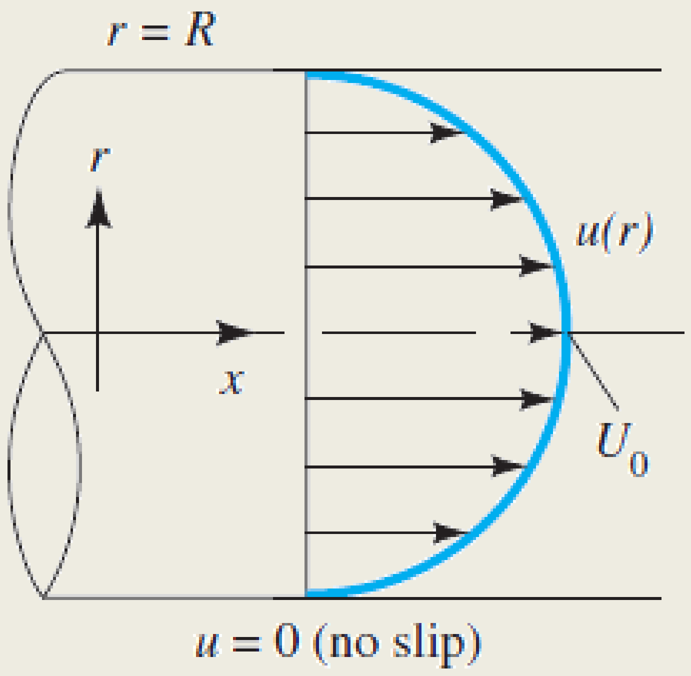
Example

$$V_{av} = \frac{1}{A}\int u\, dA = \frac{1}{\pi R^2}\int_0^R U_0\left(1 - \frac{r}{R}\right)^m 2\pi r\, dr$$

$$V_{av} = U_0 \frac{2}{(1+m)(2+m)}$$

m ≈ 1/2 → $V_{av} = 0.53\ U_0$

m ≈ 1/7 → $V_{av} = 0.82\ U_0$



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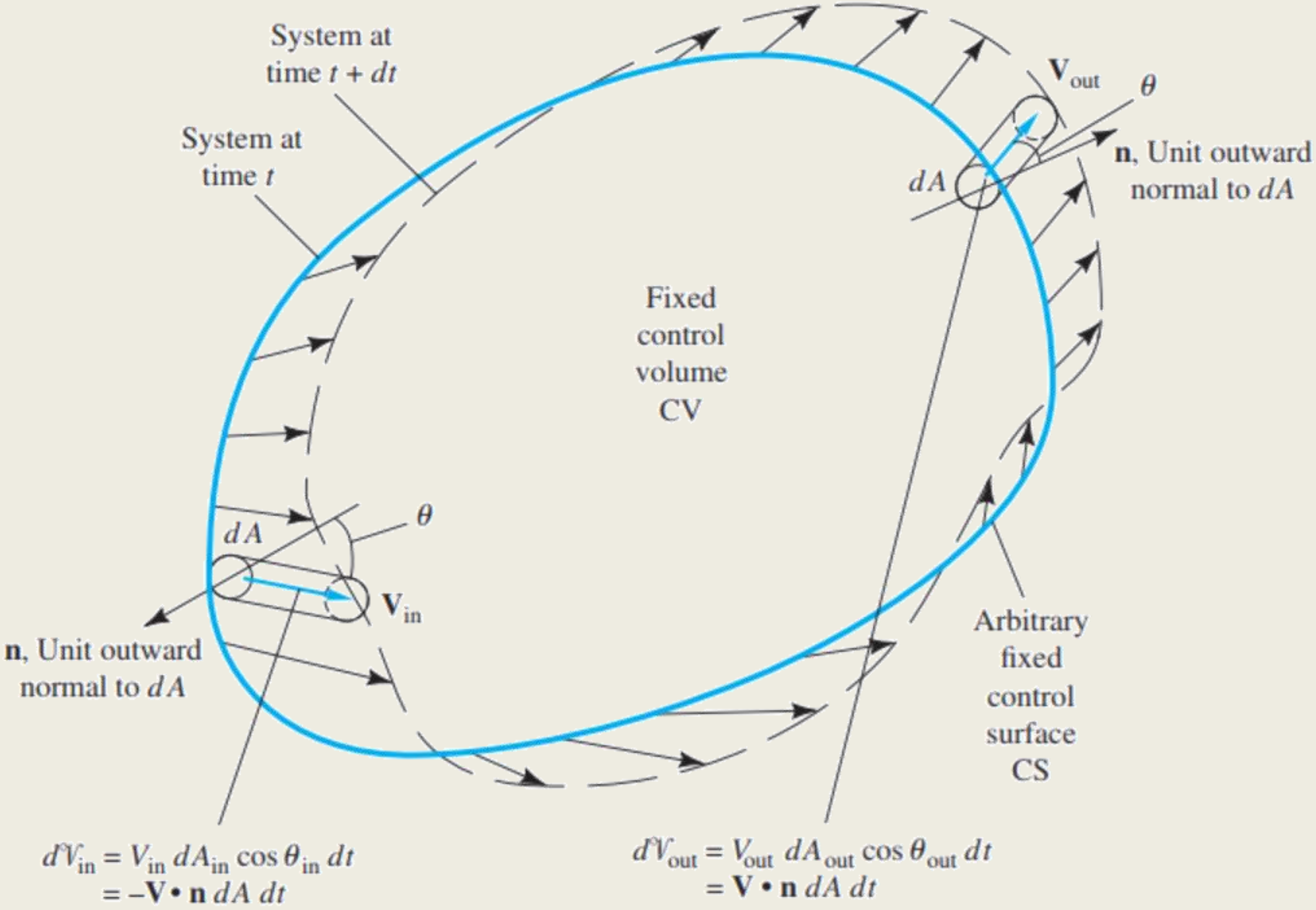
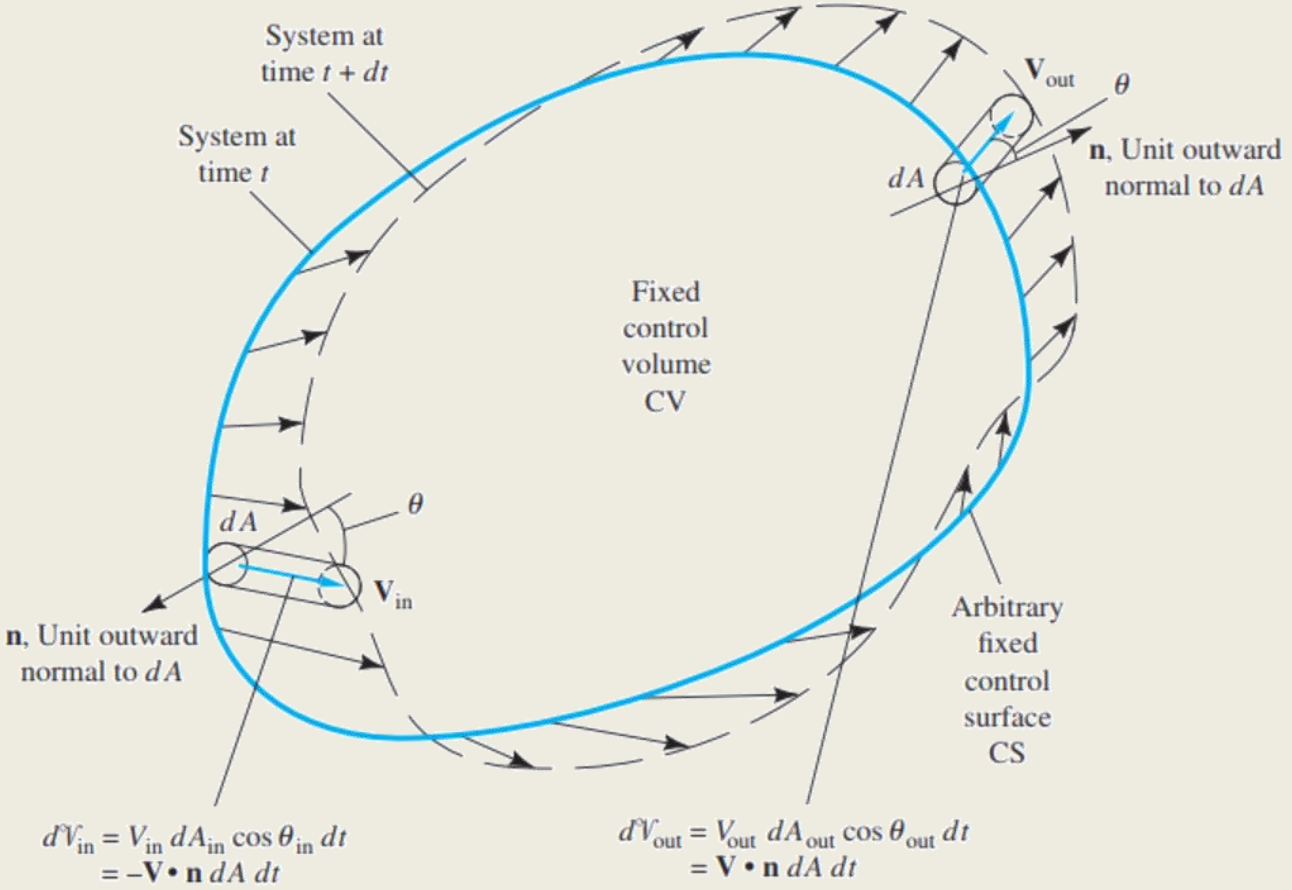
Reynolds Transform Theorem (fixed CV)



Reynolds Transform Theorem (fixed CV)

If B is the extensive property of CV like (mass, energy, momentum)

$$B_{CV} = \int_{CV} \beta \, dm = \int_{CV} \beta \rho \, d\mathcal{V} \quad \beta = \frac{dB}{dm}$$



Reynolds Transform Theorem (fixed CV)

- Change of the property B in the system is related to three property change

$$\frac{d}{dt}\left(\int_{CV} \beta \rho d\mathcal{V}\right)$$

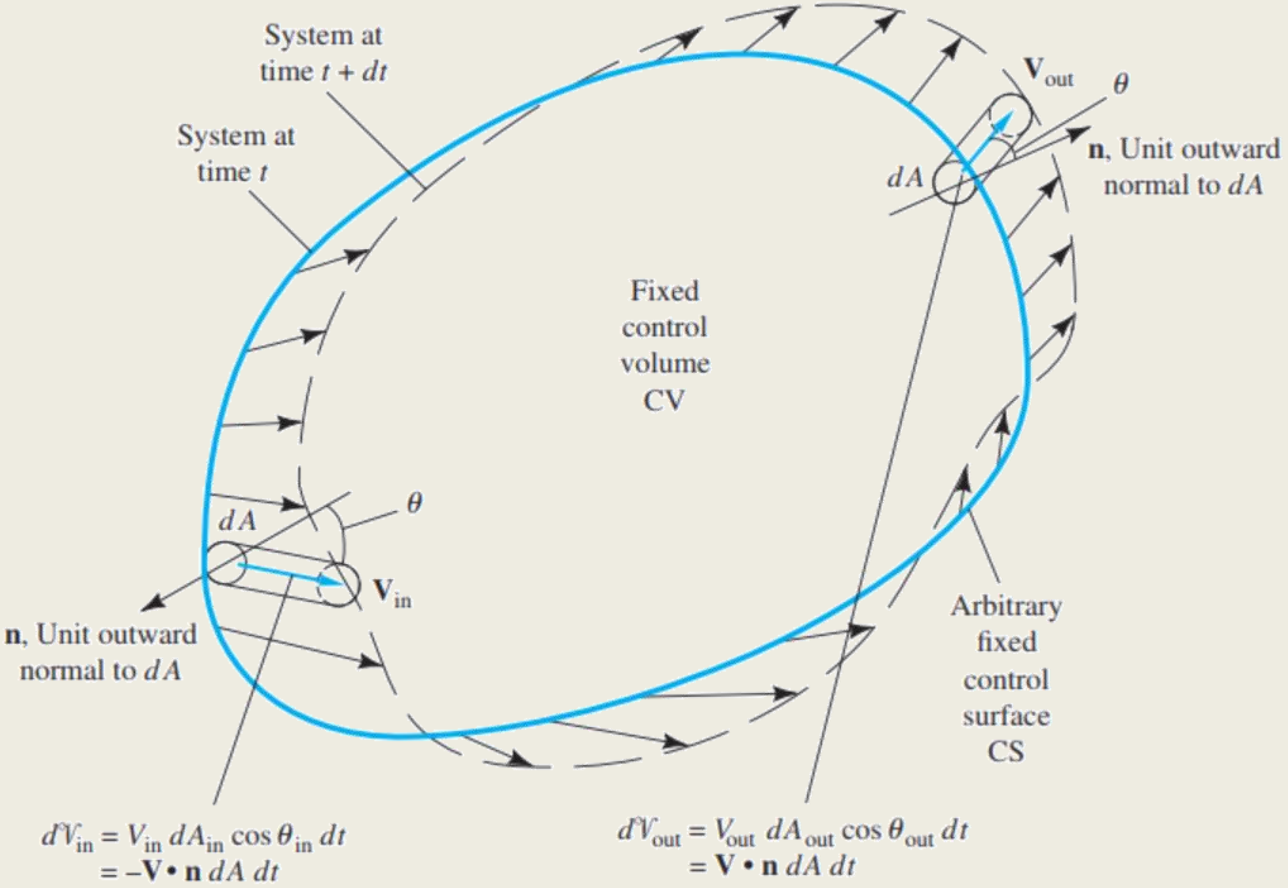
Rate of change of B in the control volume

$$\int_{CS} \beta \rho V \cos\theta \, dA_{out}$$

Rate of B out of the CV through control surface

$$\int_{CS} \beta \rho V \cos\theta \, dA_{in}$$

Rate of B into the CV through control surface



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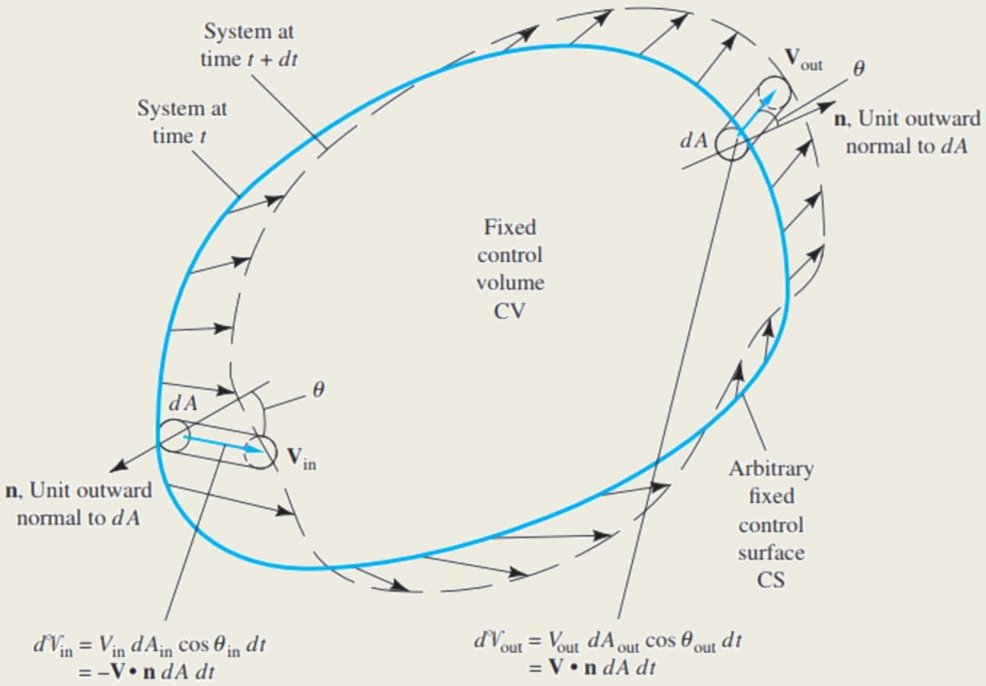
Reynolds Transform Theorem (fixed CV)

- Comparing System and CV at times t and t+dt, we can write the following:



$$\frac{d}{dt}(B_{syst}) = \frac{d}{dt}\left(\int_{CV} \beta\rho\, d\mathcal{V}\right) + \int_{CS} \beta\rho V \cos\theta\, dA_{out} - \int_{CS} \beta\rho V \cos\theta\, dA_{in}$$

Reynolds transform formula for **fixed control** volume

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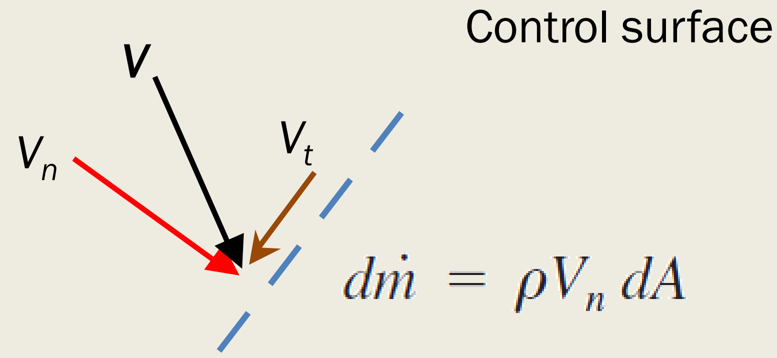
Reynolds Transform Theorem (fixed CV)

- Another form:
  - *$V\cos\theta$ is the normal component of **V** to the surface:*

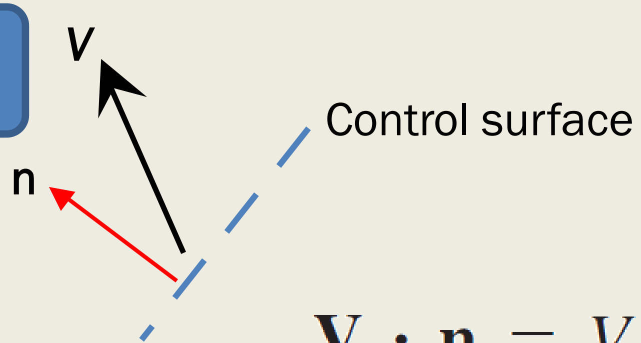
$$\text{Flow terms} = \int_{CS} \beta\rho V_n\, dA_{\text{out}} - \int_{CS} \beta\rho V_n\, dA_{\text{in}} = \int_{CS} \beta\, d\dot{m}_{\text{out}} - \int_{CS} \beta\, d\dot{m}_{\text{in}}$$



$$d\dot{m} = \rho V_n\, dA$$

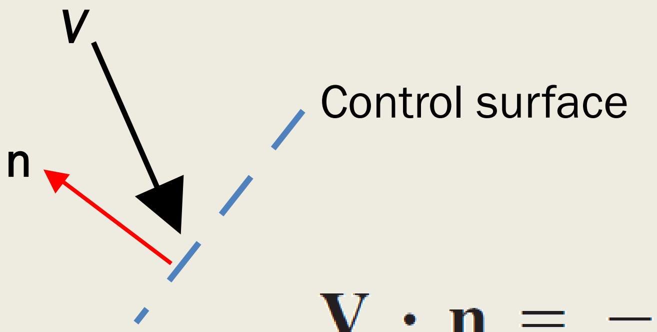
  - *In vector form (n is the outward normal unit vector of surface):*

Outlet flow



$$\mathbf{V}\cdot\mathbf{n} = V_n$$

Inlet flow



$$\mathbf{V}\cdot\mathbf{n} = -V_n$$

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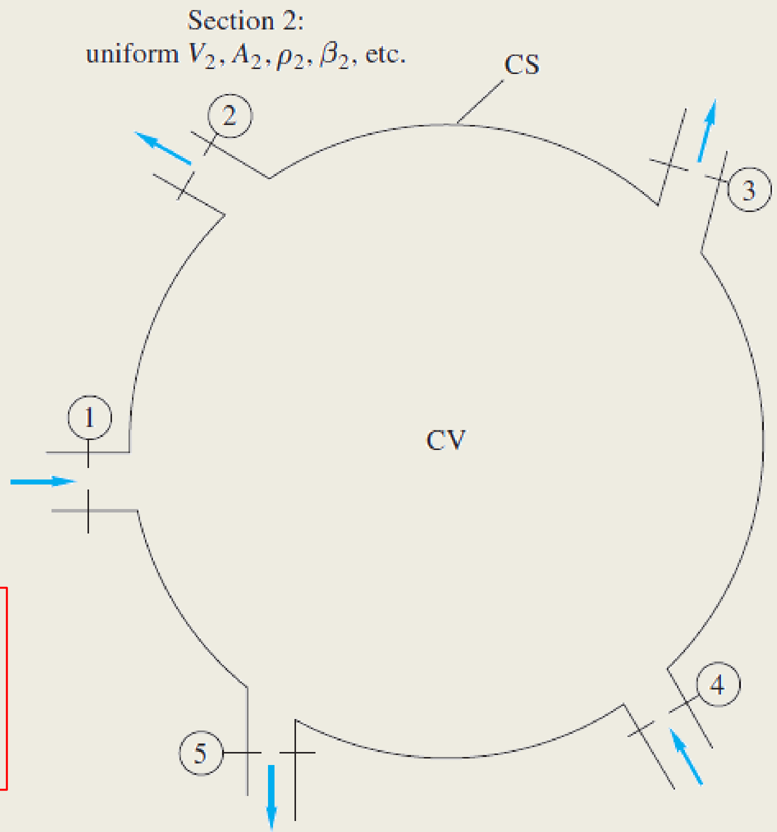
Reynolds Transform Theorem (fixed CV)

$$\frac{d}{dt}(B_{\text{syst}}) = \frac{d}{dt}\left(\int_{\text{CV}} \beta\rho \, d\mathcal{V}\right) + \int_{\text{CS}} \beta\rho(\mathbf{V}\cdot\mathbf{n})\, dA$$

Reynolds transform formula for **fixed control** volume

Simpler?

- In many practical applications the flow can be considered uniform.
- In the *uniform* flow, flow properties are constant across the integral surfaces.



$$\frac{d}{dt}(B_{syst}) = \frac{d}{dt}\left(\int_{CV} \beta \, dm\right) + \sum_{outlets} \beta_i \dot{m}_i \,|_{out} - \sum_{inlets} \beta_i \dot{m}_i \,|_{in}$$

$$\dot{m}_i = \rho_i A_i V_i$$

Conservation of Mass (Integral form)

- The property B is mass here.

$$B = m \;\; and \;\; \beta = \frac{dm}{dm} = 1$$

- Reynolds transform formula becomes:

$$\left(\frac{dm}{dt}\right)_{\text{syst}} = 0 = \frac{d}{dt}\left(\int_{\text{CV}} \rho \, d\mathcal{V}\right) + \int_{\text{CS}} \rho(\mathbf{V}_r \cdot \mathbf{n}) \, dA$$

$$\frac{d}{dt}\left(\int_{\text{CV}} \rho \, d\mathcal{V}\right) + \int_{\text{CS}} \rho(\mathbf{V}_r \cdot \mathbf{n}) \, dA = 0$$

Integral Mass Conservation Law

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Conservation of Mass (Integral form)

- Steady state flow
  - *Variation of property B with respect to t is zero:*

$$\int_{CS} \rho(\mathbf{V} \cdot \mathbf{n})\, dA = 0$$

Sum of mass flow rates of fluid out of CV is equal to the mass flow rate of fluid into the CV

- Uniform flow at inlet/outlet, steady flow:

$$\sum_i (\rho_i A_i V_i)_{\text{in}} = \sum_i (\rho_i A_i V_i)_{\text{out}}$$

$$\sum_i (\dot{m}_i)_{\text{out}} = \sum_i (\dot{m}_i)_{\text{in}}$$

Conservation of Mass (Integral form)

- Incompressible flow and the non-deforming CV:

Non-deforming CV

$$\frac{d}{dt}\left(\int_{CV} \beta\rho \, d\mathcal{V}\right) = \int_{CV} \frac{\partial}{\partial t}(\beta\rho)\, d\mathcal{V}$$

$$\frac{d}{dt}\left(\int_{CV} \rho \, d\mathcal{V}\right) + \int_{CS} \rho(\mathbf{V}_r \cdot \mathbf{n})\, dA = 0 \;\Rightarrow\; \int_{CV} \frac{\partial \rho}{\partial t} d\mathcal{V} + \int_{CS} \rho(\mathbf{V} \cdot \mathbf{n})\, dA = 0$$

Incompressible flow (ρ ≈ cte)

$$\int_{CV} \cancel{\frac{\partial \rho}{\partial t}} d\mathcal{V} + \int_{CS} \rho(\mathbf{V} \cdot \mathbf{n})\, dA = 0 \;\Rightarrow\; \int_{CS} \rho(\mathbf{V} \cdot \mathbf{n})\, dA = 0 \;\Rightarrow\; \int_{CS} (\mathbf{V} \cdot \mathbf{n})\, dA = 0$$

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Conservation of Mass (Integral form)

- Uniform flow on inlet and outlet ports:

$$\sum_i (V_i A_i)_{\text{out}} = \sum_i (V_i A_i)_{\text{in}}$$

$$\sum Q_{\text{out}} = \sum Q_{\text{in}}$$

For incompressible flow
The sum of volumetric flow into the non-deforming CV is equal to the sum of the volumetric flow out of the non-deforming CV.

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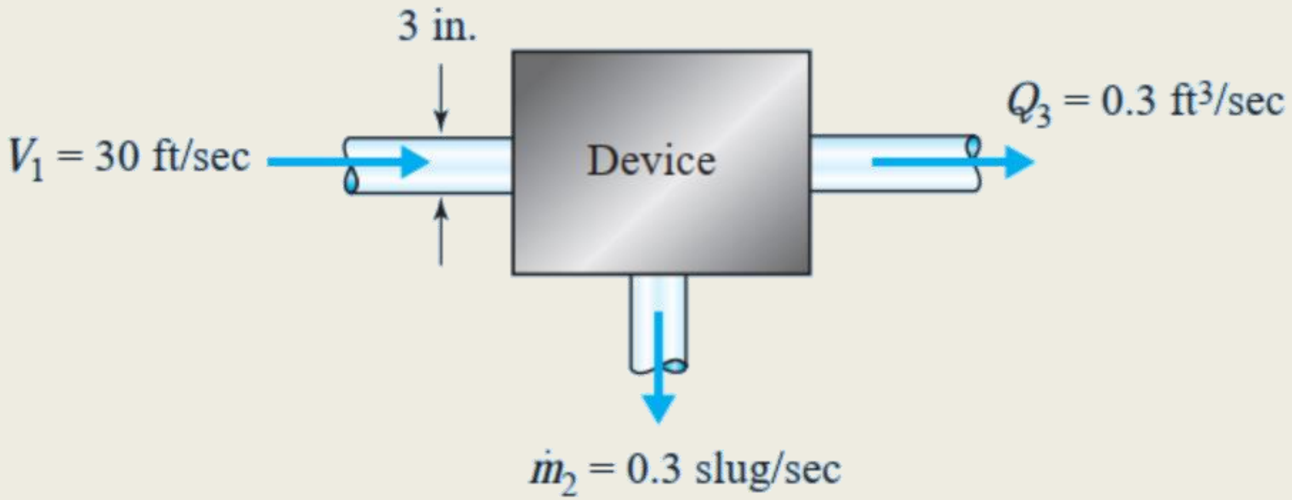
Example 1

- Water flows in and out of a device. Calculate the rate of change of the mass of water (dm/dt) in the device:



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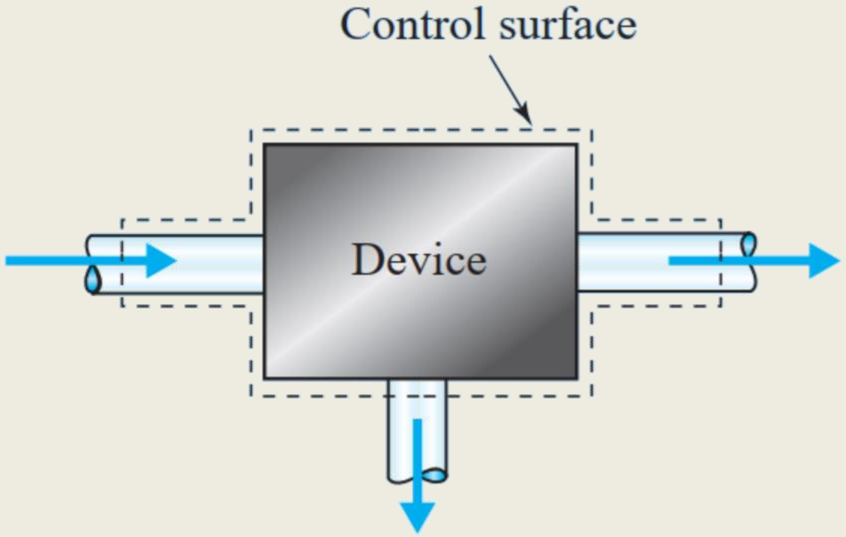
Example 1:

- Assuming uniform flow on the inlet and outlet ports:



$$0 = \frac{d}{dt}\int_{\text{c.v.}} \rho \, d\mathcal{V} + \int_{\text{c.s.}} \rho \hat{\mathbf{n}} \cdot \mathbf{V} \, dA$$

$$= \frac{dm}{dt} - \rho_1 A_1 V_1 + \rho_2 A_2 V_2 + \rho_3 A_3 V_3$$

$$0 = \frac{dm}{dt} - \rho_1 A_1 V_1 + \dot{m}_2 + \rho_3 Q_3$$

$$= \frac{dm}{dt} - 1.94 \text{ slug/ft}^3 \times \left(\pi \times \frac{1.5^2}{144}\right) \text{ft}^2 \times 30 \text{ ft/sec} + 0.3 \text{ slug/sec}$$

$$+ 1.94 \text{ slug/ft}^3 \times 0.3 \text{ ft}^3\text{/sec}$$

$$\frac{dm}{dt} = 1.975 \text{ slug/sec}$$

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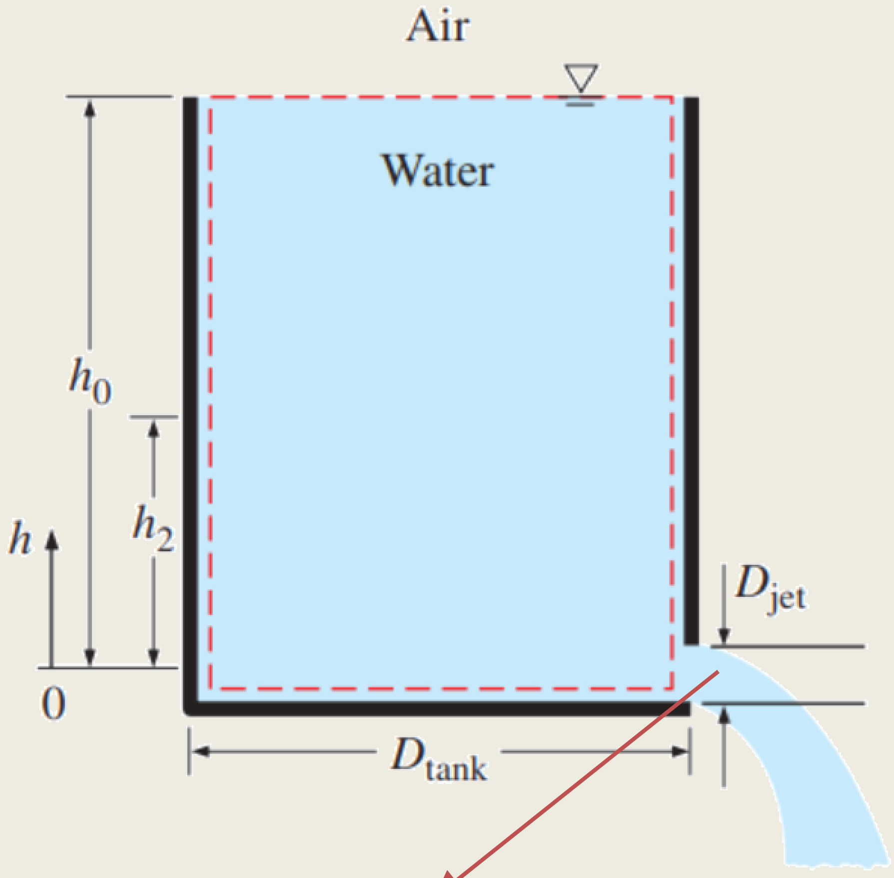
Example 2 (tank discharge)

A 4-ft-high, 3-ft-diameter cylindrical water tank is initially filled with water. Now the discharge plug near the bottom of the tank is pulled out, and a water jet whose diameter is 0.5 in streams out. The average velocity of the jet is approximated as $V = (2gh)^{0.5}$, Determine how long it takes for the water level in the tank to drop to 2 ft from the bottom.



$$V = \sqrt{2gh}$$

height of water in the tank measured from the center of the hole

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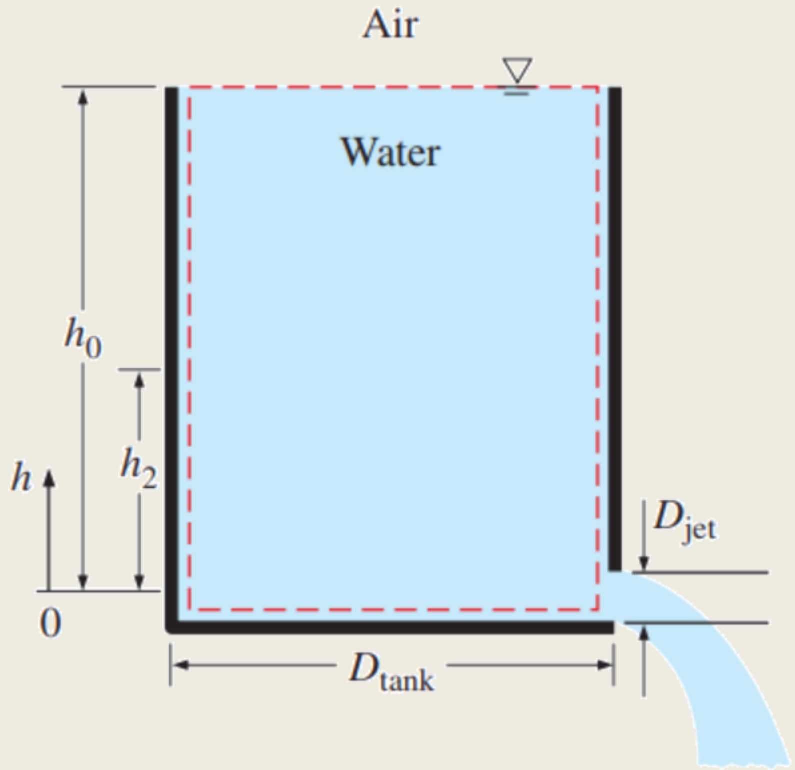
Example 2 (tank discharge)

$$\frac{d}{dt}\left(\int_{CV} \rho \, d\mathcal{V}\right) + \int_{CS} \rho(\mathbf{V}_r \cdot \mathbf{n})\, dA = 0 \quad \xrightarrow[\text{uniform flow}]{\rho = \text{cte}} \quad \frac{dV}{dt} = -\dot{Q}_{out}$$

$$A_{tank}\frac{dh}{dt} = \left(\frac{\pi}{4} D_{tank}^2\right)\frac{dh}{dt} = -V_{jet}A_{jet} = \sqrt{2gh}\left(\frac{\pi}{4} D_{jet}^2\right)$$

$$dt = -\frac{D_{\text{tank}}^2}{D_{\text{jet}}^2}\frac{dh}{\sqrt{2gh}}$$

With $t = 0$, $h = h_0$



Example 2 (tank discharge)

$$\int_0^t dt = -\frac{D_{\text{tank}}^2}{D_{\text{jet}}^2\sqrt{2g}}\int_{h_0}^{h_2}\frac{dh}{\sqrt{h}} \rightarrow t = \frac{\sqrt{h_0}-\sqrt{h_2}}{\sqrt{g/2}}\left(\frac{D_{\text{tank}}}{D_{\text{jet}}}\right)^2$$

$$t = \frac{\sqrt{4\text{ ft}}-\sqrt{2\text{ ft}}}{\sqrt{32.2/2\text{ ft/s}^2}}\left(\frac{3\times 12\text{ in}}{0.5\text{ in}}\right)^2 = 757\text{ s} = \mathbf{12.6\text{ min}}$$

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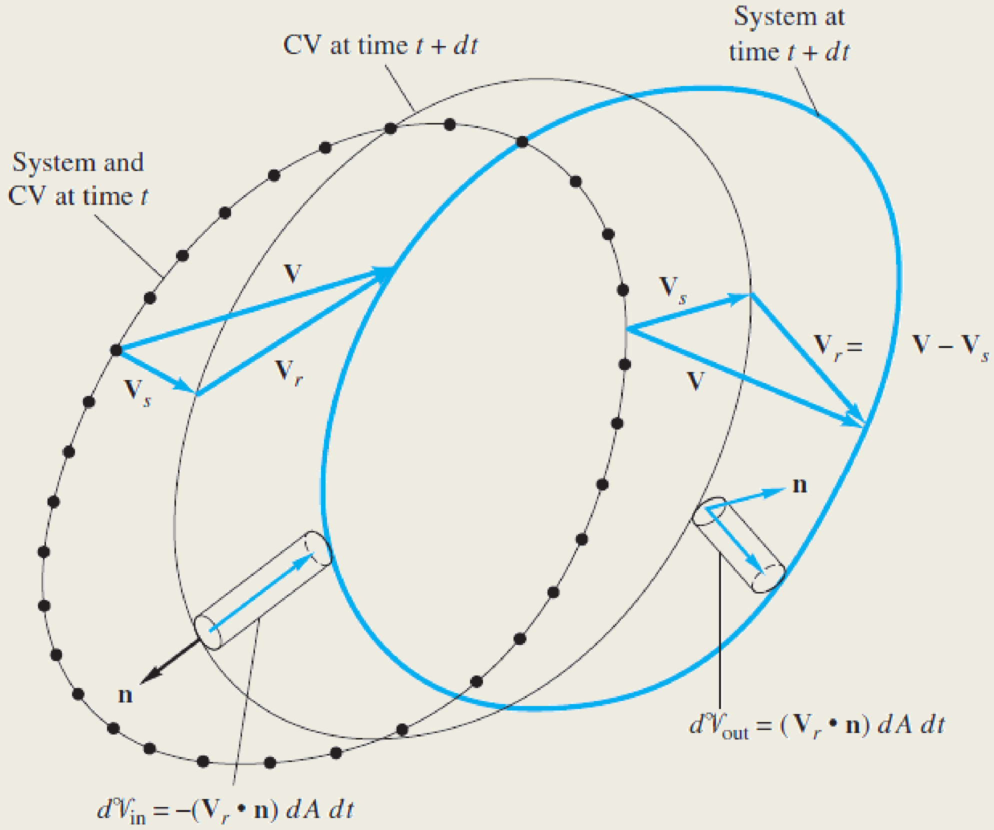
Moving control volume

- If the control volume moves at the constant velocity *Vs*, the Reynolds transform formula becomes:



$$\frac{d}{dt}(B_{syst}) = \frac{d}{dt}\left(\int_{CV} \beta\rho\, d\mathcal{V}\right) + \int_{CS} \beta\rho(\mathbf{V}_r \cdot \mathbf{n})\, dA$$

$$\mathbf{V}_r = \mathbf{V} - \mathbf{V}_s$$