MECHANICS OF FLUIDS

Lecture 6 – Integral Formulation

Linear Momentum

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Note

- All the art-work contents of this lecture are obtained from the following sources, unless otherwise stated:
  - *Fluid Mechanics, 8th edition, Frank M. White, McGraw-Hill, 2016.*
  - *Fluid Mechanics: Fundamental and Applications, 3rd edition, Yunus A. Cengel, John M. Cimbala, McGraw-Hill, 2014.*

Conservation of linear momentum

- In the linear momentum the variables B and β are:

$$\mathbf{B} = m\mathbf{V} \qquad \beta = d\mathbf{B}/dm = \mathbf{V}$$

- Based on the Newton's second law of linear motion:

$$\frac{d}{dt}(m\mathbf{V})_{\text{syst}} = \sum \mathbf{F} = \frac{d}{dt}\left(\int_{\text{CV}} \mathbf{V}\rho \, d\mathcal{V}\right) + \int_{\text{CS}} \mathbf{V}\rho(\mathbf{V}_r \cdot \mathbf{n}) \, dA$$

Sum of body & surface forces on the CV

Rate of change of momentum in the CV

Momentum flow (flux) term
Rate of momentum into or out of CV due to fluid flow

$$\frac{d}{dt}(B_{\text{syst}}) = \frac{d}{dt}\left(\int_{\text{CV}} \beta\rho \, d\mathcal{V}\right) + \int_{\text{CS}} \beta\rho(\mathbf{V}_r \cdot \mathbf{n}) \, dA$$

Conservation of linear momentum

- Note: the momentum equation is a vector equation, so it contains 3 equations for scalar variables *u*, *v*, and *w* (in Cartesian coordinates):

$$\boldsymbol{V} = (u, v, w)$$

Mass flow rate term

$$\sum F_x = \frac{d}{dt}\left(\int_{CV} u\rho \, d\mathcal{V}\right) + \int_{CS} u\overbrace{\rho(\mathbf{V}_r \cdot \mathbf{n})}\, dA$$

Sum of body and surface forces on the CV in **x-direction**

Rate of change of **x-momentum** in the CV

Rate of **x-momentum** into or out of the CV due to fluid flow

Simpler?

- For uniform flow on the inlet and outlet ports:

$$\dot{\mathbf{M}}_{CS} = \int_{sec} \mathbf{V}\rho(\mathbf{V}\cdot\mathbf{n})\,dA \xrightarrow{\text{uniform}} = \mathbf{V}_i(\rho_i V_{ni} A_i) = \dot{m}_i\mathbf{V}_i$$

- sign for inlet flow
+ sign for outflow

$$\sum \mathbf{F} = \frac{d}{dt}\left(\int_{CV} \mathbf{V}\rho\, d\mathcal{V}\right) + \sum(\dot{m}_i\mathbf{V}_i)_{out} - \sum(\dot{m}_i\mathbf{V}_i)_{in}$$

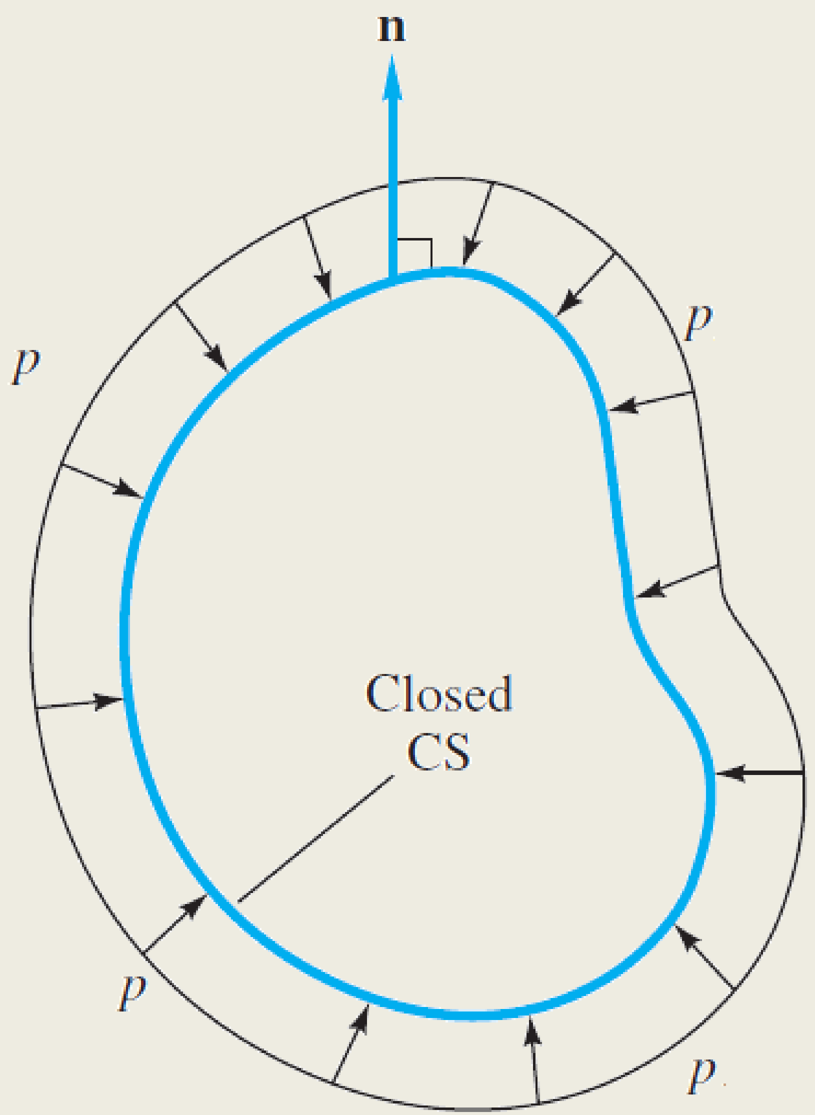
$$\sum F = ?$$

- *Weight of the fluid due to gravity on the body (CV)*
- *Viscous force on the control surface (CS), will be discussed in later chapters.*
- ***Pressure force** on the control surface (CS)*
- *Other external forces on the CV/CS*

Net pressure force on the CS

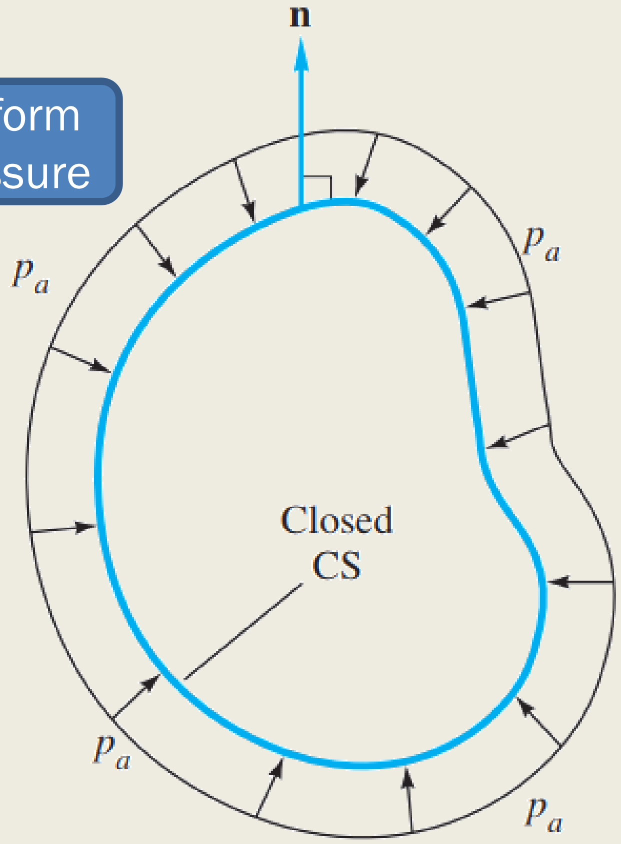
$$\mathbf{F}_{\text{press}} = \int_{\text{CS}} p(-\mathbf{n})\, dA$$

Pressure force in **inward** and the normal vector is **outward**



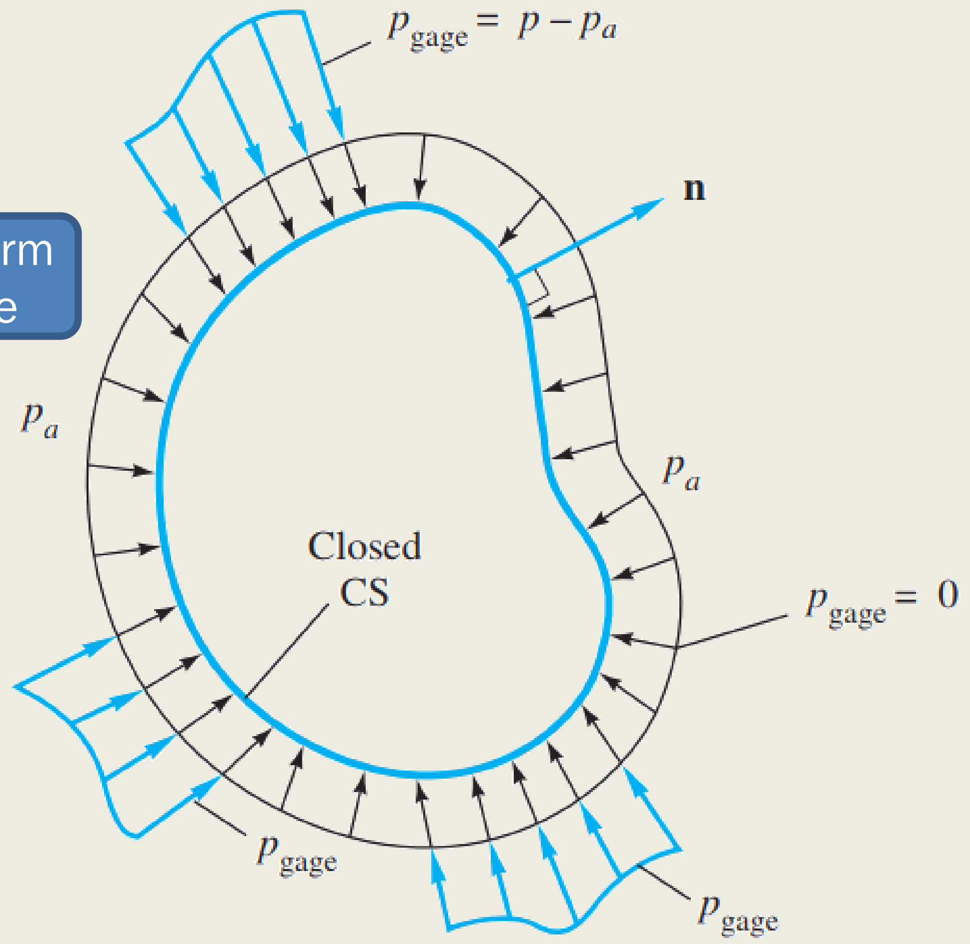
Net pressure force on the CS

Uniform pressure



$$F_{press} = \int p_a(-\mathbf{n})dA = -p_a \int \mathbf{n}\, dA \equiv 0$$

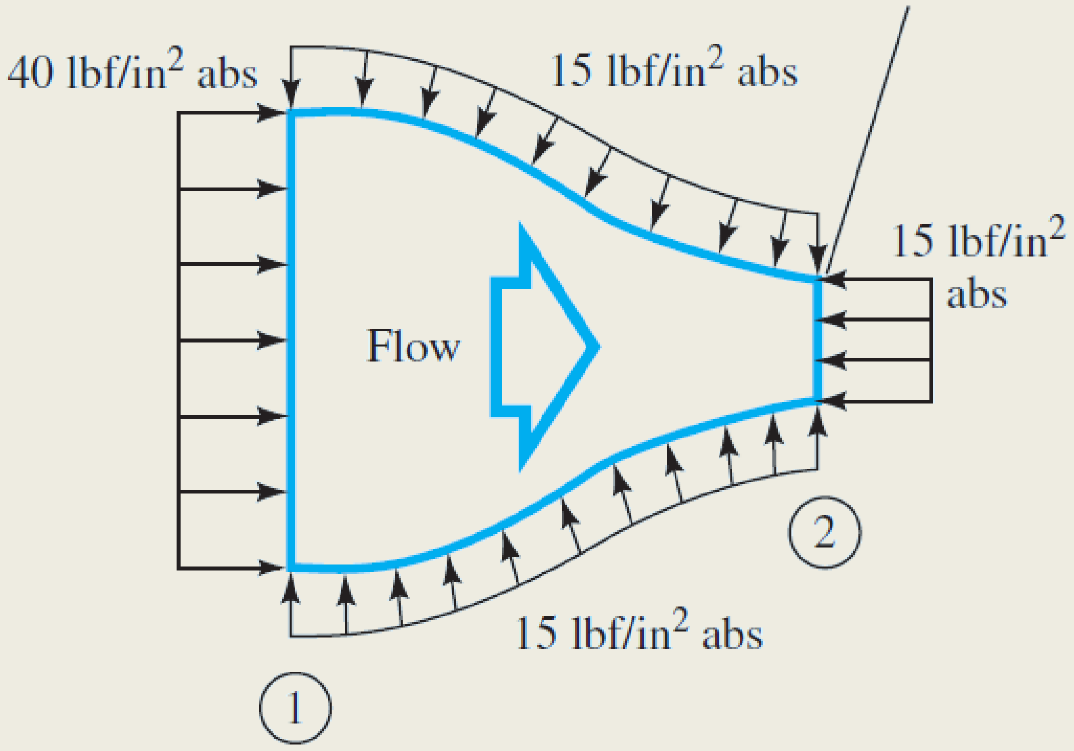
Non-uniform pressure



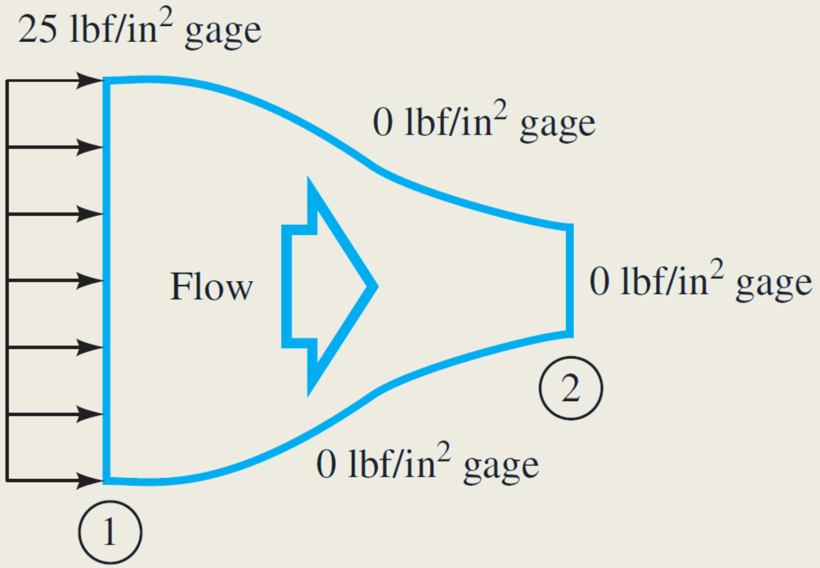
$$F_{press} = \int_{CS} (p - p_a)(-\mathbf{n})dA = \int_{CS} p_{gage}(-\mathbf{n})\, dA$$

Example 1

A control volume of a nozzle section has surface pressures of 40 psia at section 1 and atmospheric pressure of 15 psia at section 2 and on the external rounded part of the nozzle. Compute the net pressure force if D_1 = 3 in and D_2 = 1 in.



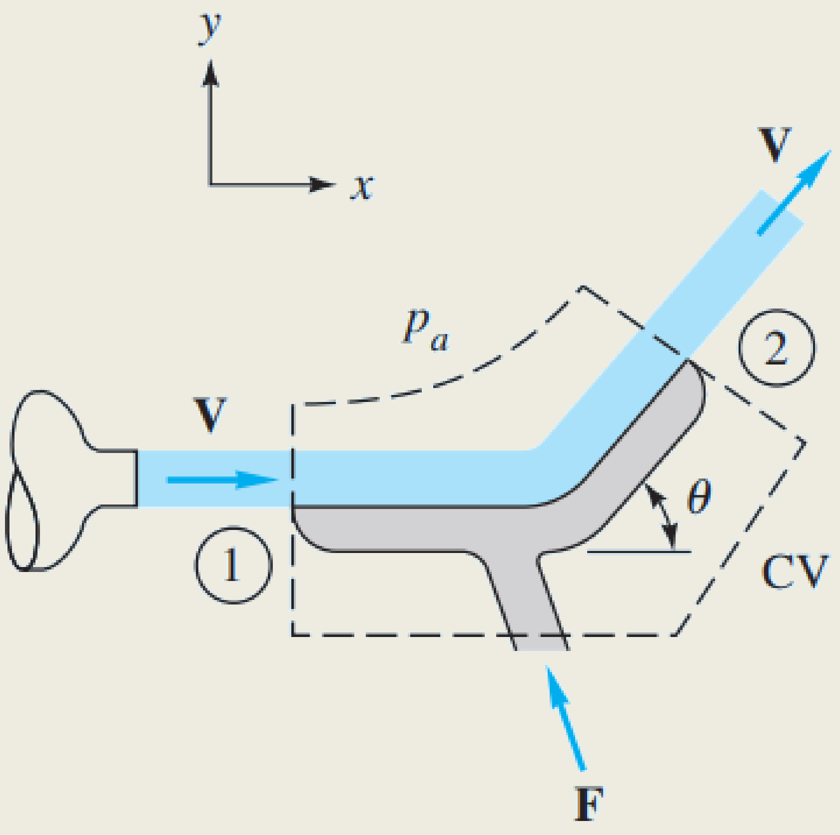
Example 1



$$\mathbf{F}_{\text{press}} = p_{\text{gage},1}\,(-\mathbf{n})_1\,A_1 = \left(25\,\frac{\text{lbf}}{\text{in}^2}\right)\left[-(-\mathbf{i})\right]\left[\frac{\pi}{4}\,(3\text{ in})^2\right] = 177\mathbf{i}\text{ lbf}$$

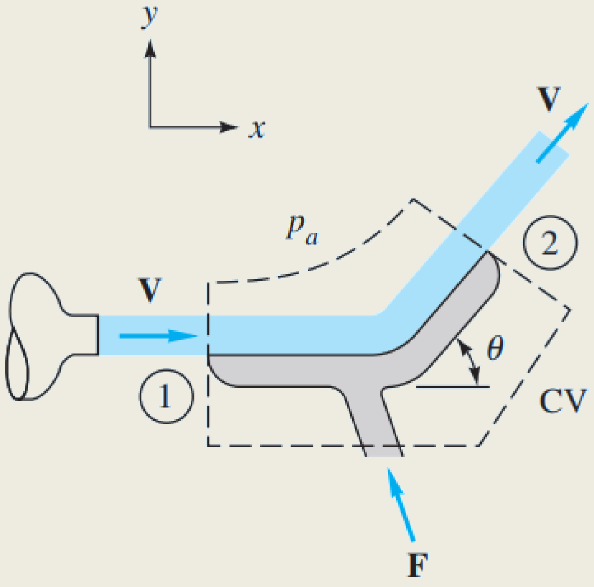
Example 2

A fixed vane turns a water jet of area *A* through an angle θ without changing its velocity magnitude. The flow is steady, pressure is p_a everywhere, and friction on the vane is negligible. Find the components *Fx* and *Fy* of the applied vane force.



Example 2

(1) Friction is negligible, (2) Pressure force is zero in atmosphere, (3) the flow is steady, (4) neglecting the weight of fluid and vane



$$\sum \mathbf{F} = \frac{d}{dt}\left(\int_{CV} \mathbf{V}\rho\, d\mathcal{V}\right) + \sum(\dot{m}_i\mathbf{V}_i)_{out} - \sum(\dot{m}_i\mathbf{V}_i)_{in}$$

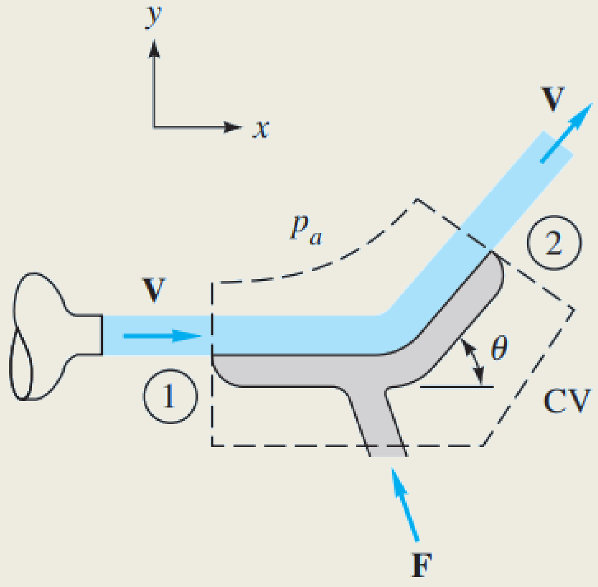
Net pressure force = 0
Weight is neglected
Friction is neglected

$$\mathbf{F}_{vane} = \dot{m}_2\mathbf{V}_2 - \dot{m}_1\mathbf{V}_1$$

Continuity at steady state gives: $\dot{m}_1 = \dot{m}_2 \Rightarrow \rho V_1 A_1 = \rho V_2 A_2$

Proving that $|\boldsymbol{V}_1| = |\boldsymbol{V}_2| = V$, since area of jet is unchanged.

Example 2



x-component and y-component of momentum equation

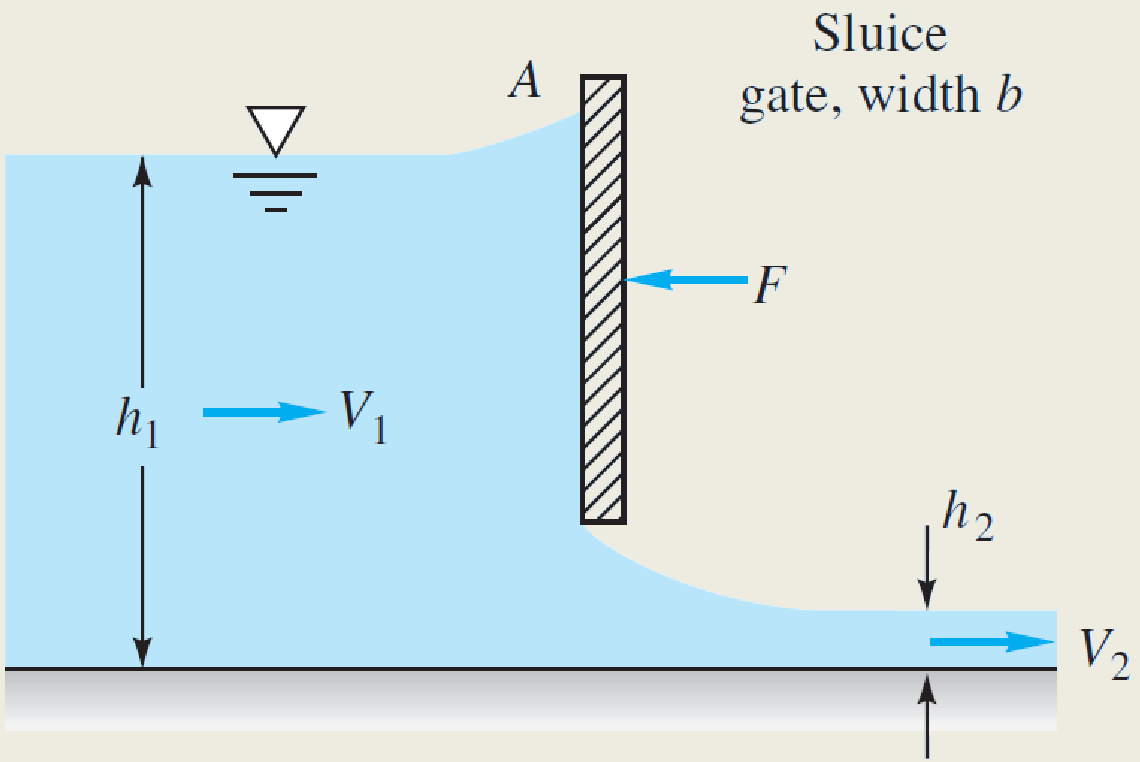
$$F_x = \dot{m}V(\cos\theta - 1) \qquad F_y = \dot{m}V\sin\theta$$

Total force

$$F = (F_x^2 + F_y^2)^{1/2} = \dot{m}V[\sin^2\theta + (\cos\theta - 1)^2]^{1/2} = 2\dot{m}V\sin\frac{\theta}{2}$$

Example 3

The sluice gate controls flow in open channels. At sections 1 and 2, the flow is uniform and the pressure is hydrostatic. Neglecting bottom friction and atmospheric pressure, derive a formula for the horizontal force *F* required to hold the gate. Express your final formula in terms of the inlet velocity V_1, eliminating V_2.



Example 3

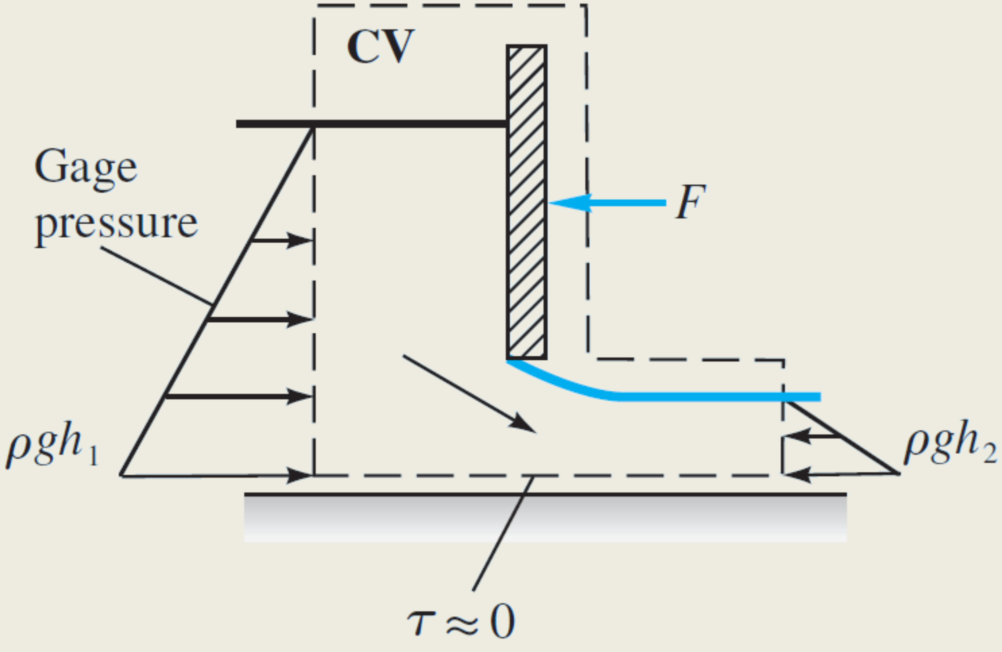
- For uniform flow, steady flow, incompressible flow

Mass balance

$$\frac{d}{dt}\left(\int_{CV} \rho \, d\mathcal{V}\right) + \int_{CS} \rho(\mathbf{V}_r \cdot \mathbf{n}) \, dA = 0$$

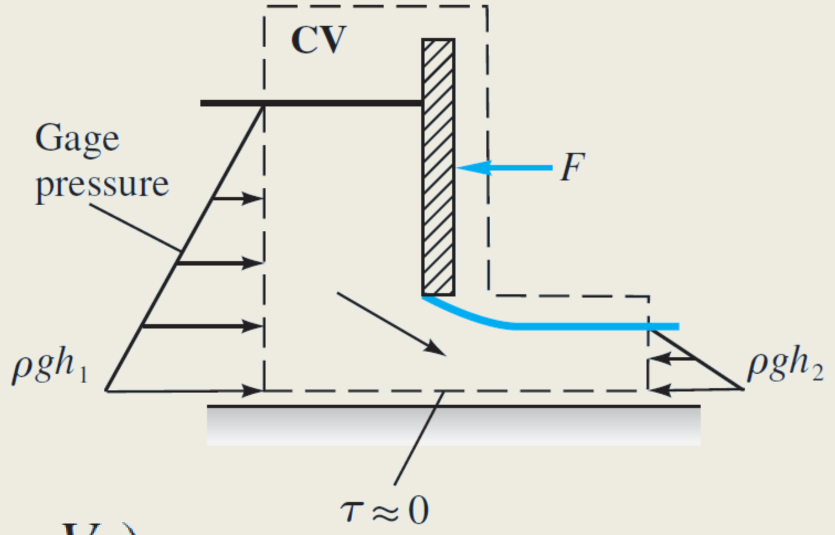
$$\rho V_1 h_1 b = \rho V_2 h_2 b$$

$$V_2 = V_1(h_1/h_2)$$



Example 3

- Linear momentum balance, for uniform, steady, and incompressible flow



$$\sum F_x = -F_{\text{gate}} + \frac{\rho}{2} g h_1 (h_1 b) - \frac{\rho}{2} g h_2 (h_2 b) = \dot{m}\,(V_2 - V_1)$$

$$V_2 = V_1 (h_1 / h_2)$$

$$F_{\text{gate}} = \frac{\rho}{2} g b h_1^2 \left[1 - \left(\frac{h_2}{h_1} \right)^2 \right] - \rho h_1 b V_1^2 \left(\frac{h_1}{h_2} - 1 \right)$$

Momentum flux correction factor

- Assuming uniform flow in the inlets and outlets and using algebraic relation $\dot{m}V_{av}$ as the momentum into or out of the CV is erroneous.
- Instead, $\dot{m}\xi V_{av}$ is used, where ξ is the correction factor for momentum

$$\sum \mathbf{F} = \frac{d}{dt}\left(\int_{CV} \mathbf{V}\rho \, d\mathcal{V}\right) + \sum (\xi_i \dot{m}_i \mathbf{V}_i)_{out} - \sum (\xi_i \dot{m}_i \mathbf{V}_i)_{in}$$

Momentum correction factor

- Assuming the flow is 1-D in x-direction

$$\rho \int u^2 dA = \zeta \dot{m} V_{av} = \zeta \rho A V_{av}^2$$

$$\zeta = \frac{1}{A} \int \left(\frac{u}{V_{av}} \right)^2 dA$$

Momentum correction factor relation

Momentum correction factor

- Laminar flow

$$u = U_0\left(1 - \frac{r^2}{R^2}\right) \Rightarrow \zeta = \frac{4}{3}$$

- Turbulent flow

$$u \approx U_0\left(1 - \frac{r}{R}\right)^m \quad \frac{1}{9} \le m \le \frac{1}{5} \Rightarrow \zeta = \frac{(1+m)^2(2+m)^2}{2(1+2m)(2+2m)}$$

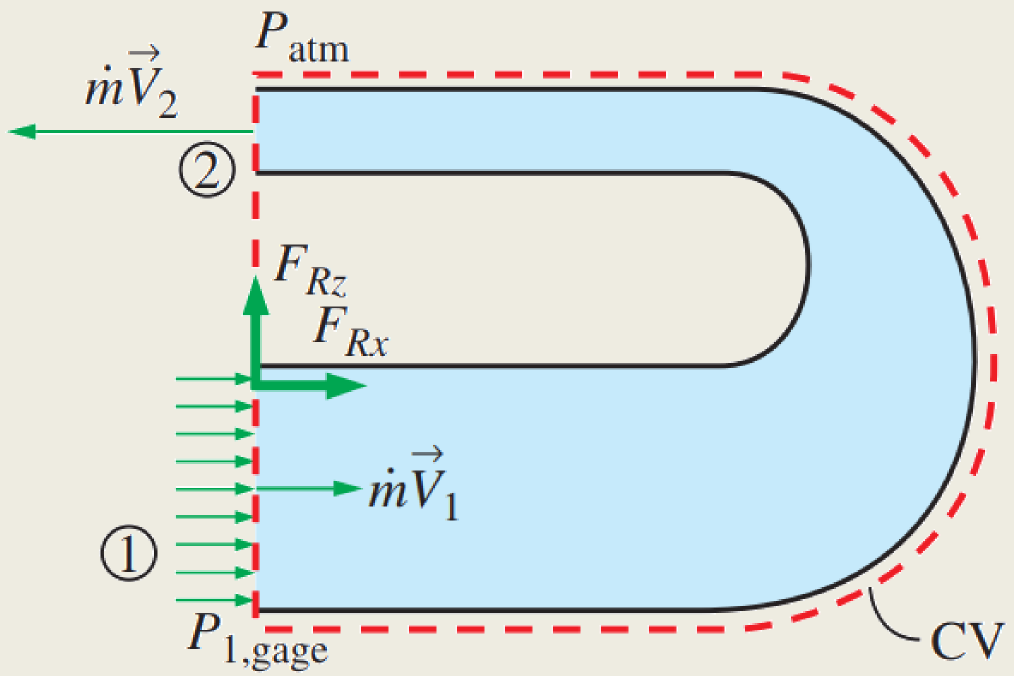
m	$\frac{1}{5}$	$\frac{1}{6}$	$\frac{1}{7}$	$\frac{1}{8}$	$\frac{1}{9}$
ζ	1.037	1.027	1.020	1.016	1.013

Turbulent vs. laminar flow

Flow direction

Example 4:

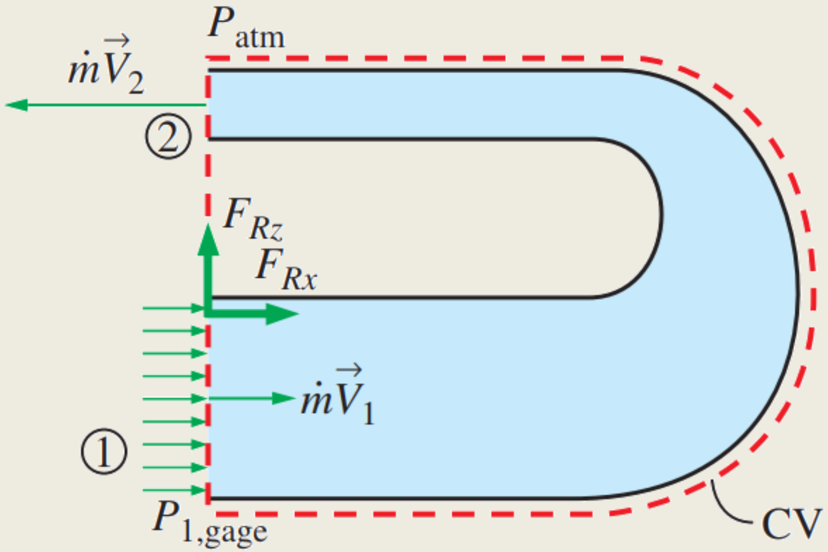
The reversing elbow shown in the figure takes water and makes a 180° U-turn before it is discharged to atmosphere. The elevation difference between the centers of the inlet and the exit sections is 0.3 m. The inlet mass flow rate of water is 14 kg/s and the cross-sectional area is 113 cm^2 at the inlet and 7 cm^2 at the outlet. The gage pressure at the inlet is 202.2 kPa. Neglection the weight of elbow and water in the elbow, determine the force needed to hold the elbow in place.



Example 4:

Assumptions: The flow is steady, the weight of the elbow and the water in it is negligible, the flow is turbulent and fully developed at both the inlet and outlet of the control volume, and we take the momentum-flux correction factor to be $\beta = 1.03$ (as a conservative estimate) at both the inlet and the outlet.

Properties: the density of water is 1000 kg/s.



Example 4:

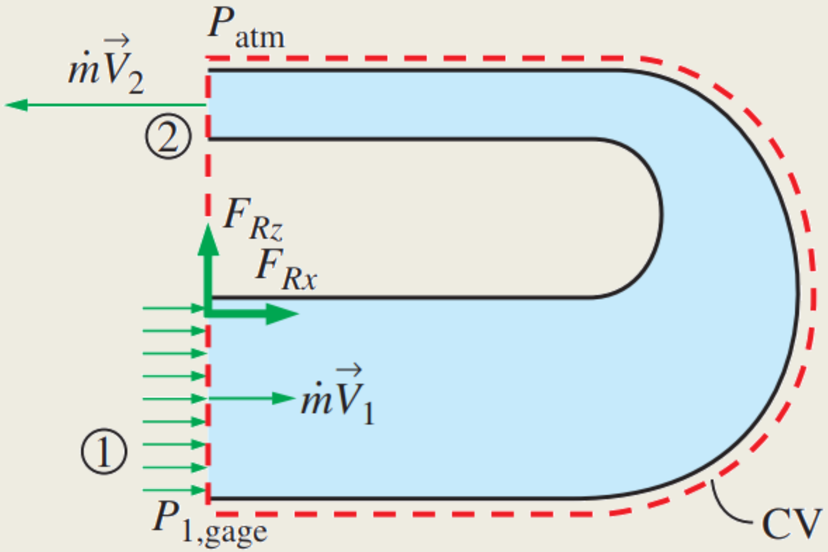
- Since the flow is steady:

$$\dot{m}_1 = \dot{m}_2 = 14\frac{kg}{s}$$

- Average velocity at the inlet and outlet:

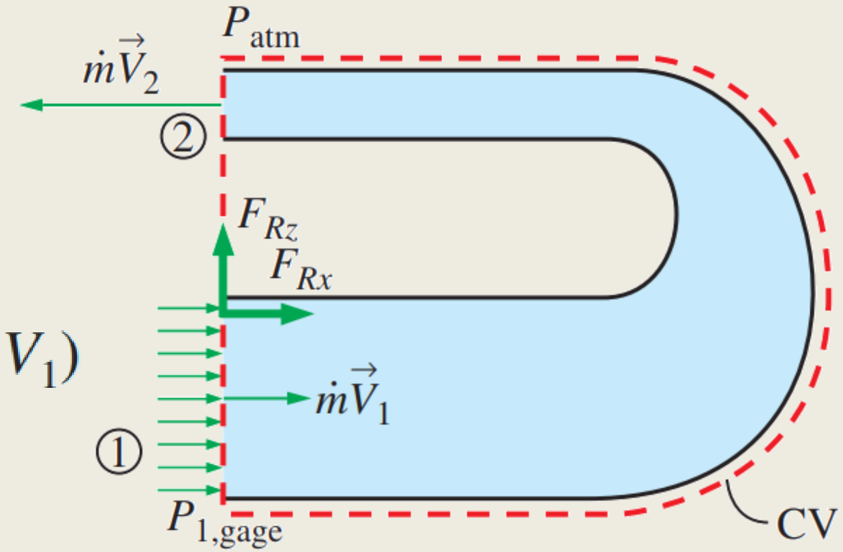
$$V_1 = \frac{\dot{m}}{\rho A_1} = \frac{14\text{ kg/s}}{(1000\text{ kg/m}^3)(0.0113\text{ m}^2)} = 1.24\text{ m/s}$$

$$V_2 = \frac{\dot{m}}{\rho A_2} = \frac{14\text{ kg/s}}{(1000\text{ kg/m}^3)(7\times 10^{-4}\text{ m}^2)} = 20.0\text{ m/s}$$



Example 4:

- Horizontal force: x-momentum equation with correction factor.



$$F_{Rx} + P_{1,\text{ gage}}A_1 = \beta_2\dot{m}(-V_2) - \beta_1\dot{m}V_1 = -\beta\dot{m}(V_2 + V_1)$$

$$F_{Rx} = -\beta\dot{m}(V_2 + V_1) - P_{1,\text{ gage}}A_1$$

$$= -(1.03)(14\text{ kg/s})[(20 + 1.24)\text{ m/s}]\left(\frac{1\text{ N}}{1\text{ kg}\cdot\text{m/s}^2}\right) - (202{,}200\text{ N/m}^2)(0.0113\text{ m}^2)$$

$$= -306 - 2285 = \mathbf{-2591\text{ N}}$$

- Vertical force is zero.