MECHANICS OF FLUIDS

Lecture 7 – Bernoulli Equation

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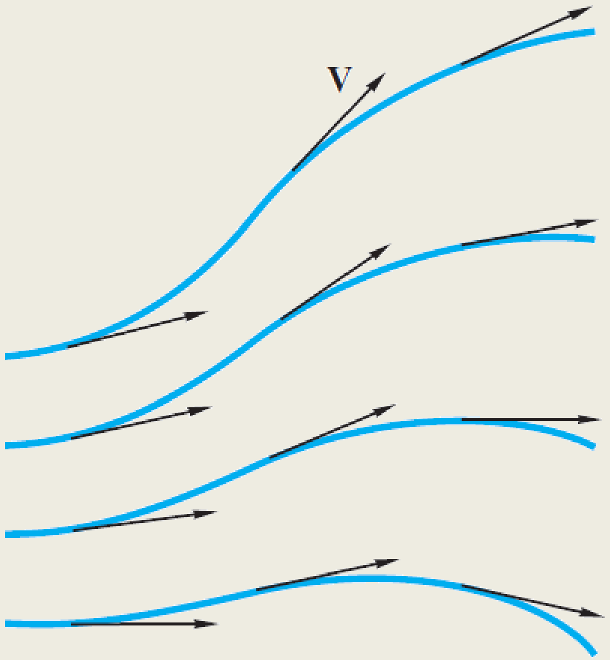
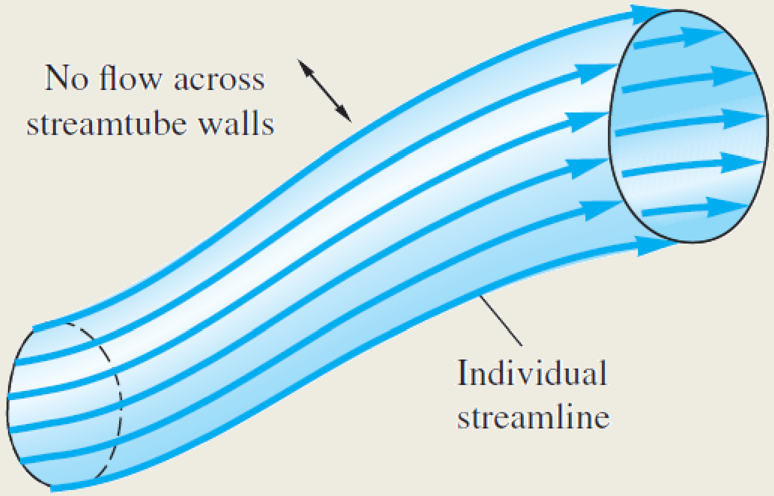
Note

- All the art-work contents of this lecture are obtained from the following sources, unless otherwise stated:
  - *Fluid Mechanics, 8th edition, Frank M. White, McGraw-Hill, 2016.*
  - *Fluid Mechanics: Fundamental and Applications, 3rd edition, Yunus A. Cengel, John M. Cimbala, McGraw-Hill, 2014.*

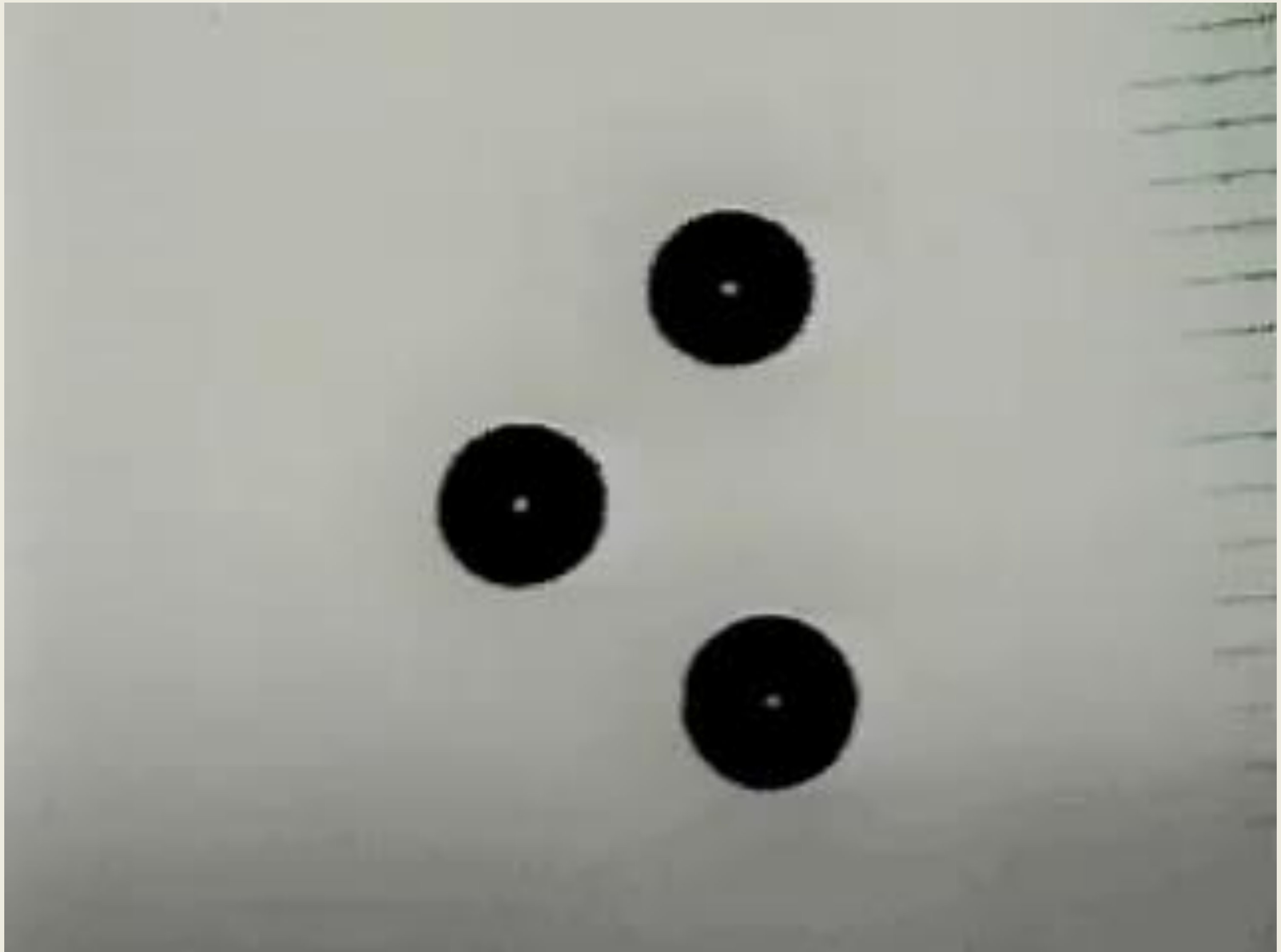
What is a streamline?

- It is a line that is tangent to velocity vector everywhere in the fluid for a given instant.



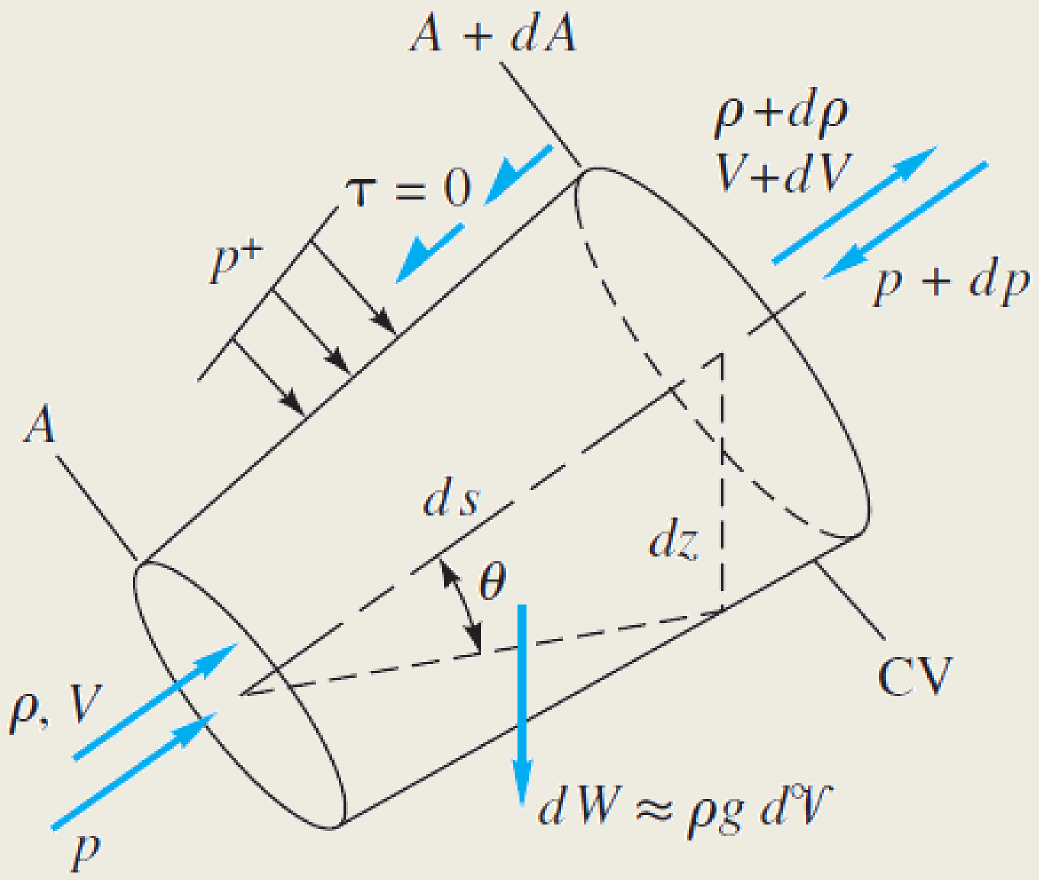


Streamline visualization



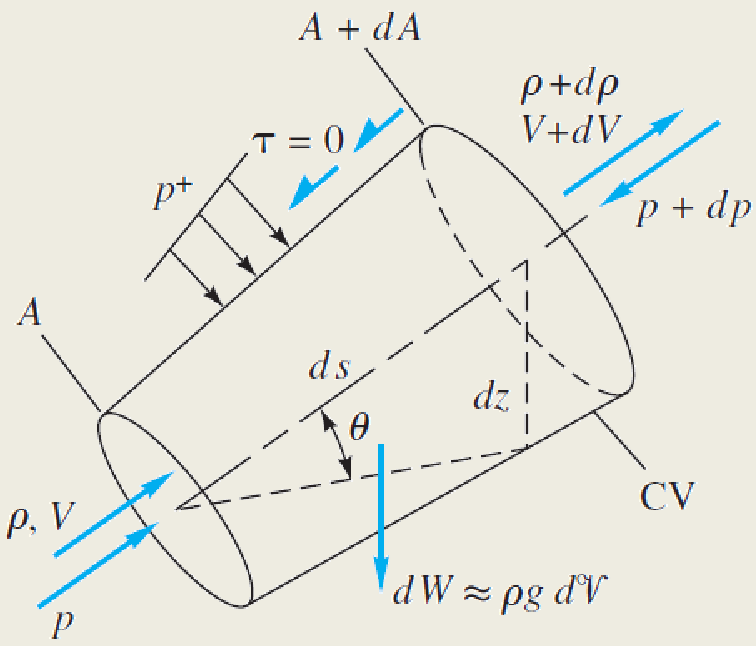
Frictionless flow analysis of a streamline

- Consider a differential fluid element along a streamline with the length *ds*



Frictionless flow analysis of a streamline

- Considering a uniform flow at the inlet and outlet, the mass balance equation for this differential element is:



$$\frac{d}{dt}\left(\int_{CV} \rho \, d\mathcal{V}\right) + \dot{m}_{out} - \dot{m}_{in} = 0 \approx \frac{\partial \rho}{\partial t} d\mathcal{V} + d\dot{m}$$

$$d\mathcal{V} \approx A \; ds$$

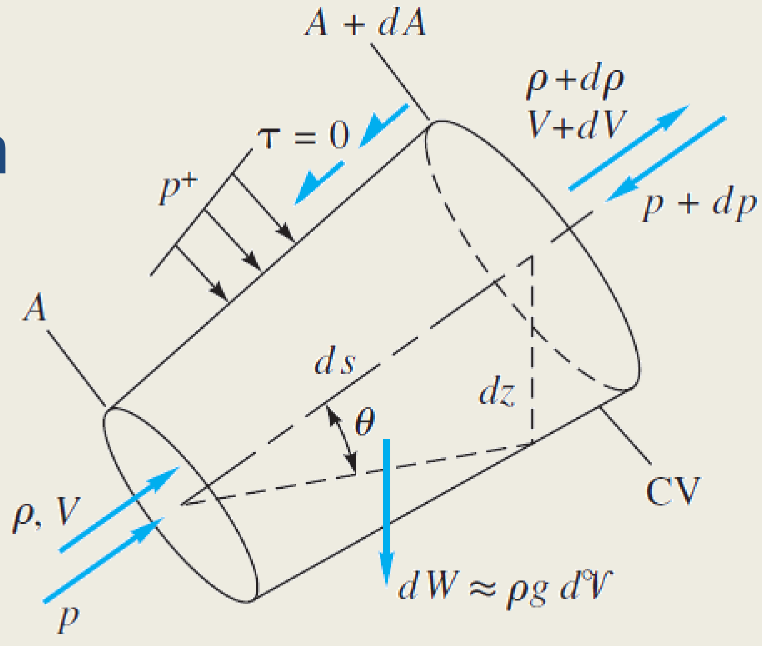
$$\dot{m} = \rho A V$$

$$d\dot{m} = d(\rho A V) = -\frac{\partial \rho}{\partial t} A \; ds$$

$$\frac{\partial \rho}{\partial t} A \, ds + d\dot{m} = 0 \qquad (*)$$

Frictionless flow analysis of a streamline

- Linear momentum balance along the stream direction, s, and neglecting the shear force (friction):



$$\sum dF_s = \frac{d}{dt}\left(\int_{CV} V\rho \, d\mathcal{V}\right) + (\dot{m}V)_{out} - (\dot{m}V)_{in} \approx \frac{\partial}{\partial t}(\rho V)\, A\, ds + d(\dot{m}V)$$

(**)

- Forces acting on control volume in s-direction (we are neglecting friction):

Net pressure force, we use gage pressure

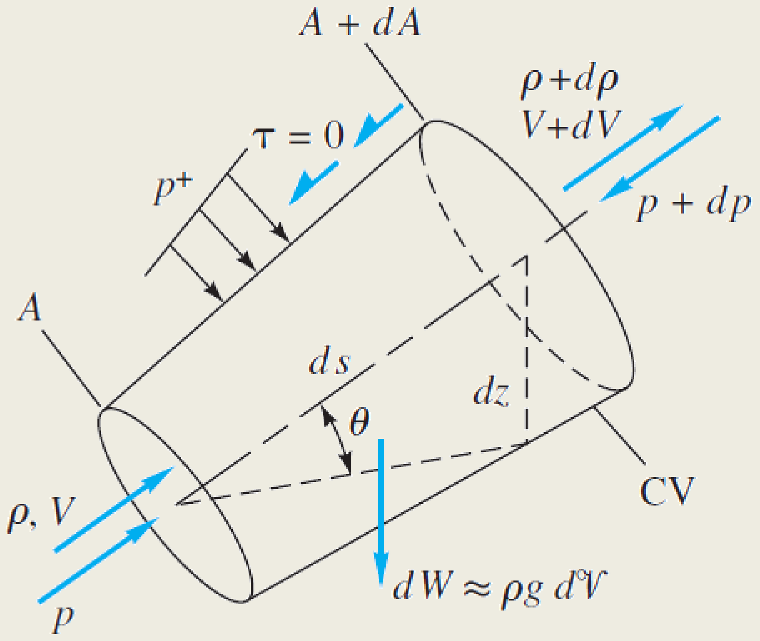
$$dF_{s,\text{press}} = \tfrac{1}{2}\, dp\, dA - dp(A + dA) \approx -A\, dp$$

(***)

Fluid weight in the control volume in s-direction

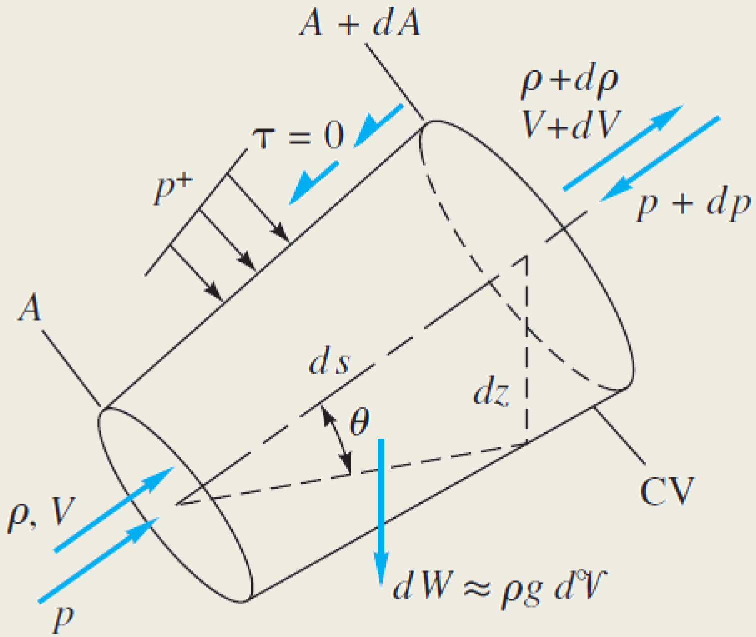
$$dF_{s,\text{grav}} = -dW \sin\theta = -\gamma A\, ds \sin\theta = -\gamma A\, dz$$

(****)



Frictionless flow analysis of a streamline

- Substituting (***) and (****) into (**) and expanding the differential operator:



$$\sum dF_s = -\gamma A\,dz - A\,dp = \frac{\partial}{\partial t}(\rho V)\,A\,ds + d(\dot{m}V)$$

$$= \frac{\partial \rho}{\partial t} VA\,ds + \frac{\partial V}{\partial t}\rho A\,ds + \dot{m}\,dV + V\,d\dot{m}$$

$$V\left(\frac{\partial \rho}{\partial t} A\,ds + d\dot{m}\right) = 0 \quad (*)$$

$\div \rho V$

$$\frac{\partial V}{\partial t}ds + \frac{dp}{\rho} + V\,dV + g\,dz = 0$$

Differential form of Bernoulli Equation along the stream line

Frictionless flow analysis of a streamline

- Integration of the above equation between two points:

$$\int_1^2 \frac{\partial V}{\partial t}\, ds + \int_1^2 \frac{dp}{\rho} + \frac{1}{2}(V_2^2 - V_1^2) + g(z_2 - z_1) = 0$$

Integral form of Bernoulli Equation along a stream line

Bernoulli Equation

- For steady and incompressible (constant density) flow:

$$\int_1^2 \frac{\partial V}{\partial t} ds + \int_1^2 \frac{dp}{\rho} + \frac{1}{2}(V_2^2 - V_1^2) + g(z_2 - z_1) = 0$$

0 Constant density

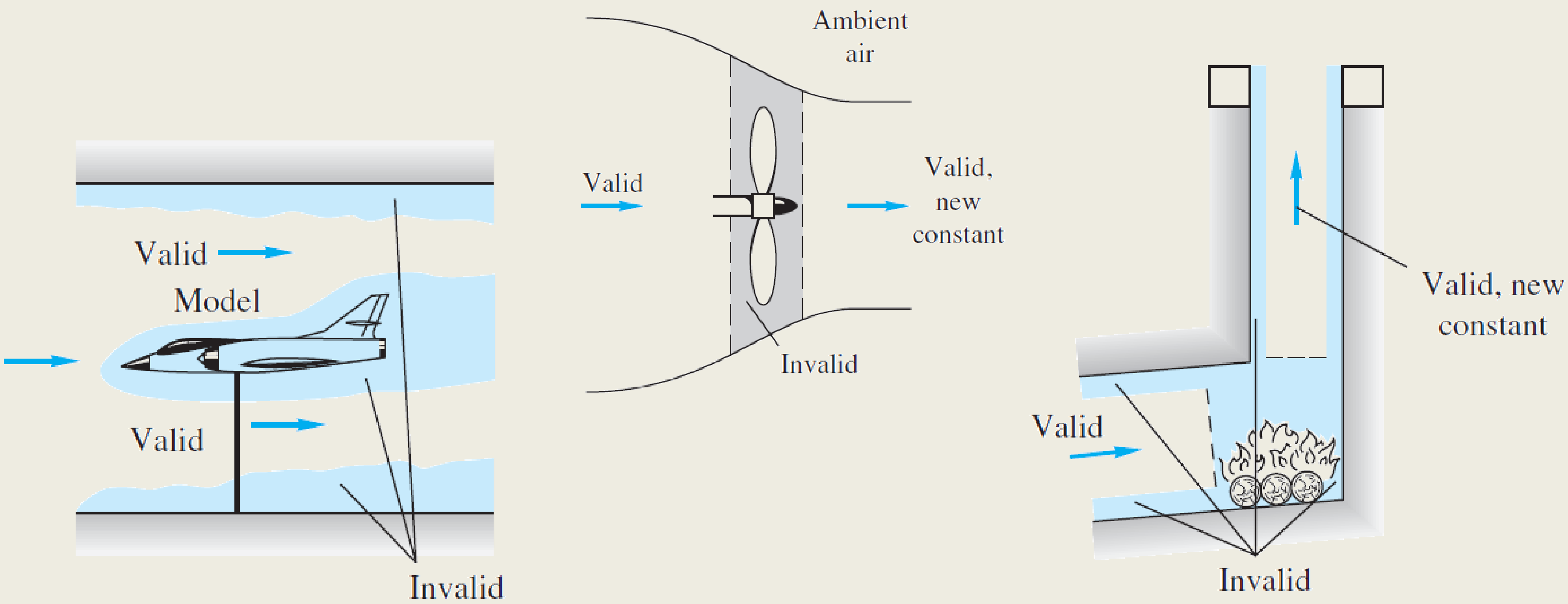
$$\frac{p_2 - p_1}{\rho} + \frac{1}{2}(V_2^2 - V_1^2) + g(z_2 - z_1) = 0$$

or

$$\frac{p_1}{\rho} + \frac{1}{2}V_1^2 + gz_1 = \frac{p_2}{\rho} + \frac{1}{2}V_2^2 + gz_2 = \text{const}$$

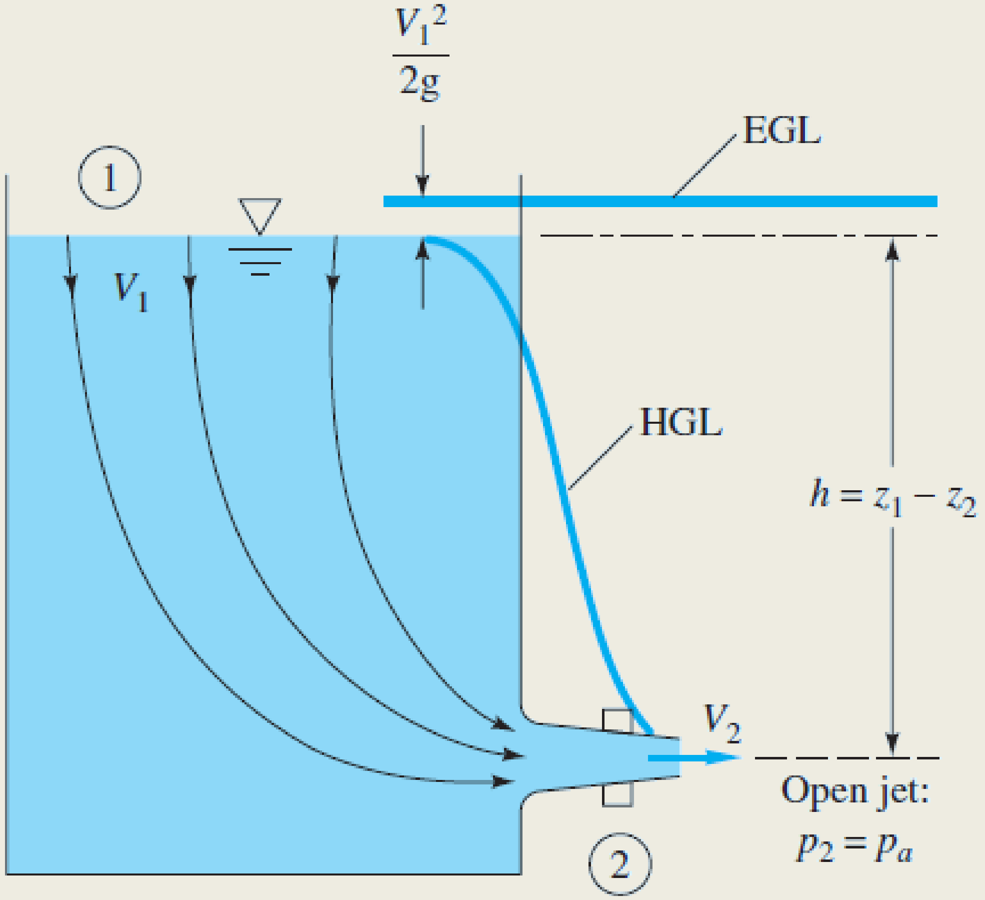
When is Bernoulli equation valid?

- Assumptions?
  - *Flow along a streamline*
    Steady flow
  - *Frictionless flow*
  - *Incompressible flow*
  - *No energy exchange between points 1 and 2*



Example: free discharge

- Find an expression for the discharge velocity (V_2) for steady, frictionless flow.



Example: free discharge

- Mass balance for incompressible flow and assuming no change in height:

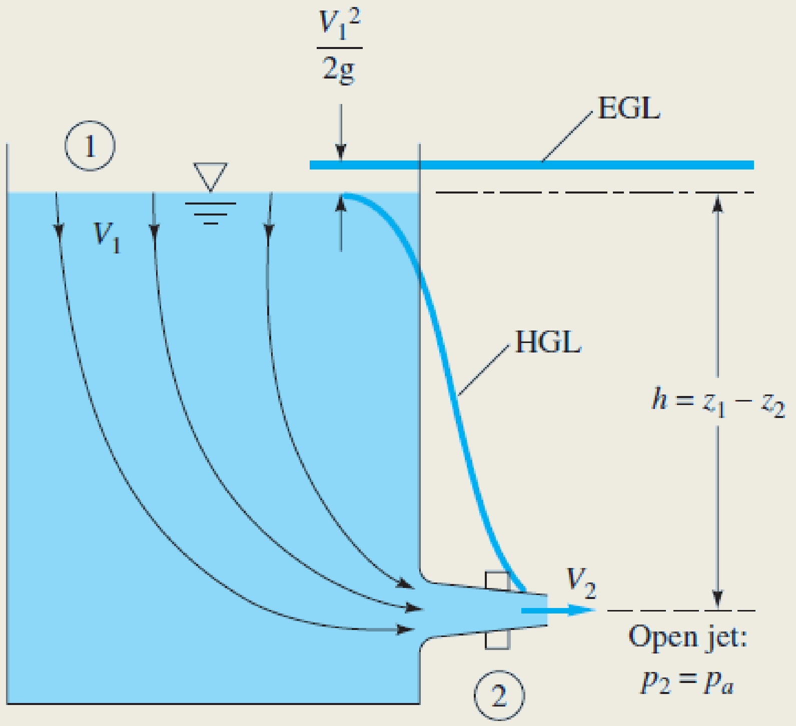
$$A_1V_1 = A_2V_2 \qquad (*)$$

- Bernoulli relation between points 1 and 2:

$$\frac{p_1}{\rho} + \frac{1}{2}V_1^2 + gz_1 = \frac{p_2}{\rho} + \frac{1}{2}V_2^2 + gz_2$$

- $p_1 = p_2$, since both points are exposed to atmosphere:

$$V_2^2 - V_1^2 = 2g(z_1 - z_2) = 2gh$$



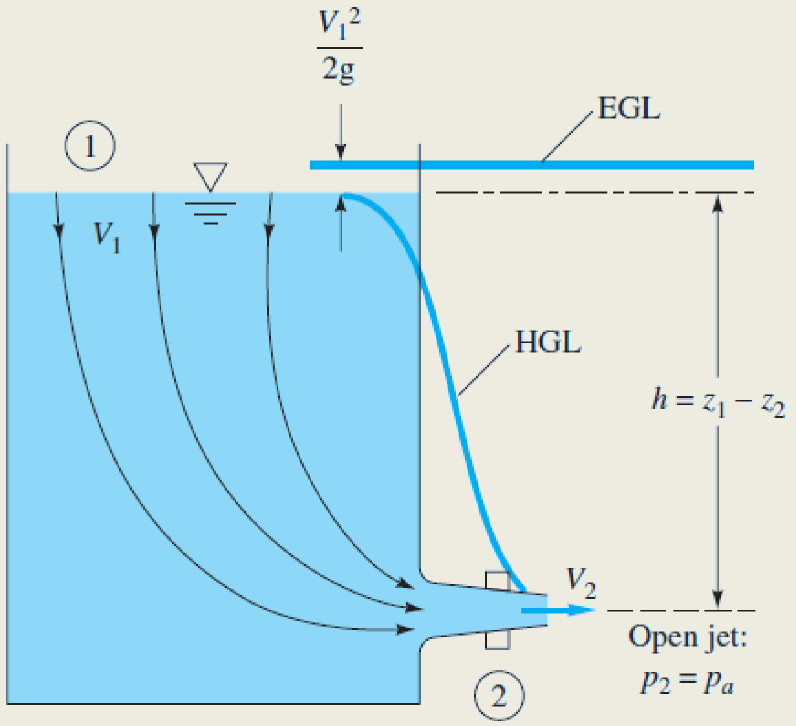
Example: free discharge

(*) ⇒ $$V_2^2 = \frac{2gh}{1 - A_2^2/A_1^2}$$

$A_1 >> A_2$ ⇒ $$V_2 \approx (2gh)^{1/2}$$

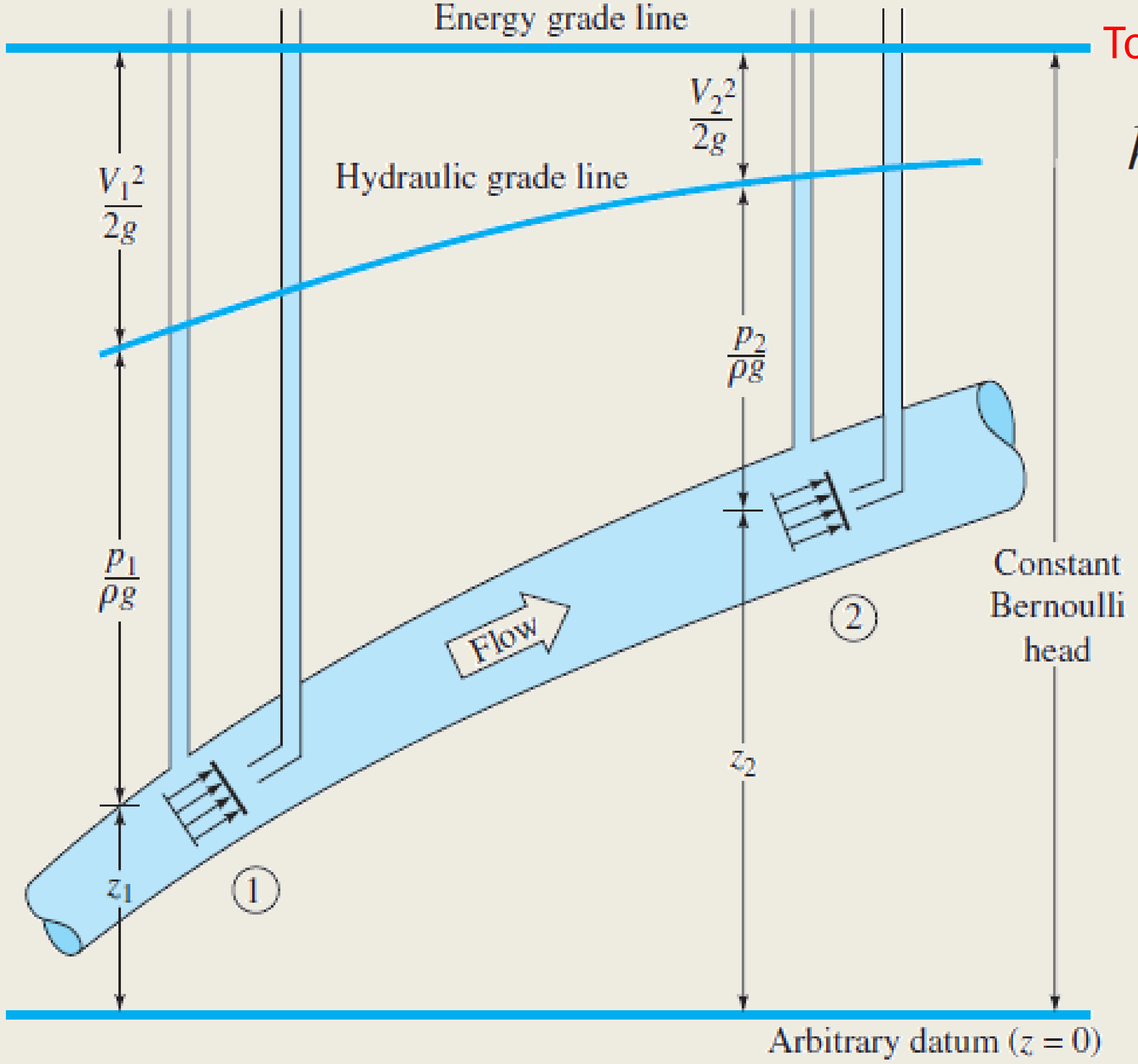
$$(V_2)_{av} = \frac{Q}{A_2} = c_d(2gh)^{1/2}$$

To account for all the friction losses and non-uniform flow in the jet we introduce discharge coefficient.



Free discharge

Hydraulic and energy grade lines



Total head

$$h_0 = z + p/\gamma + V^2/(2g)$$

Frictionless flow with no
Heat transfer or shaft work

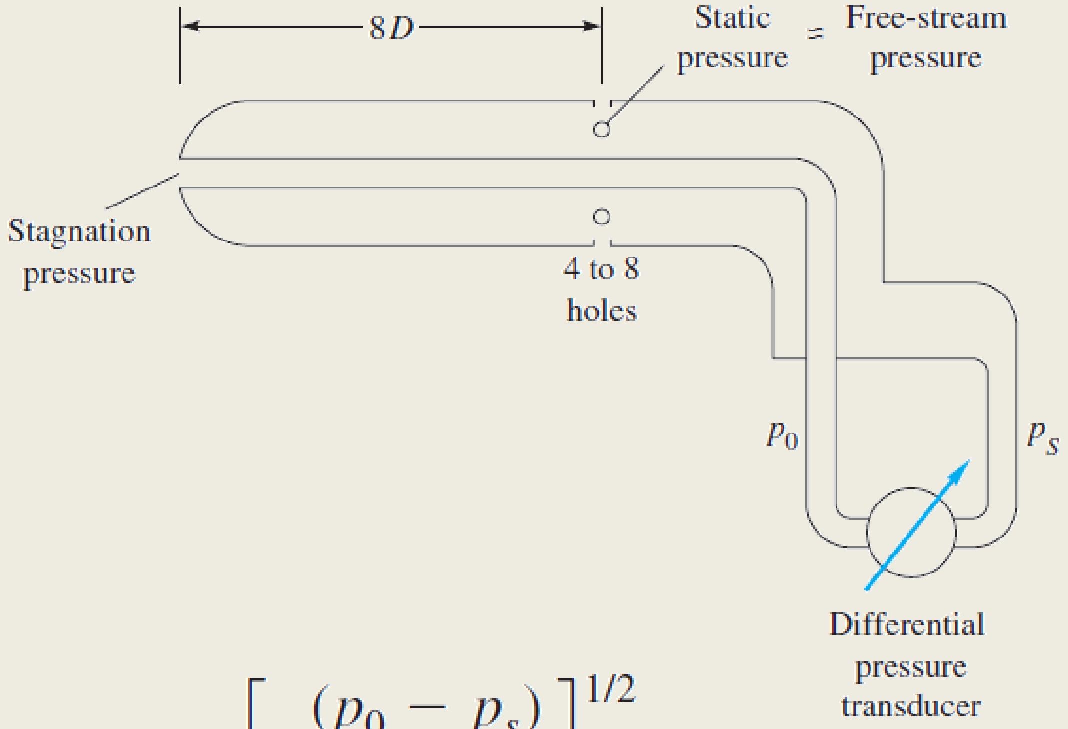
Stagnation of flow

- Consider a horizontal stream line where the elevation is zero, if we reversibly bring the flow to rest (zero velocity), the Bernoulli equation between state 1 and state 0 becomes:

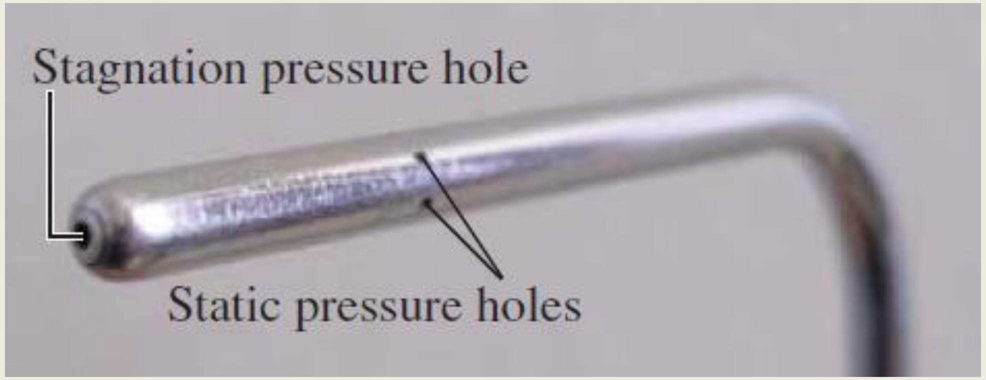
$$p_1 + \frac{1}{2}\rho V_1^2 = p_o = \text{constant}$$

Static pressure — Dynamic pressure — Stagnant pressure

Pitot probe



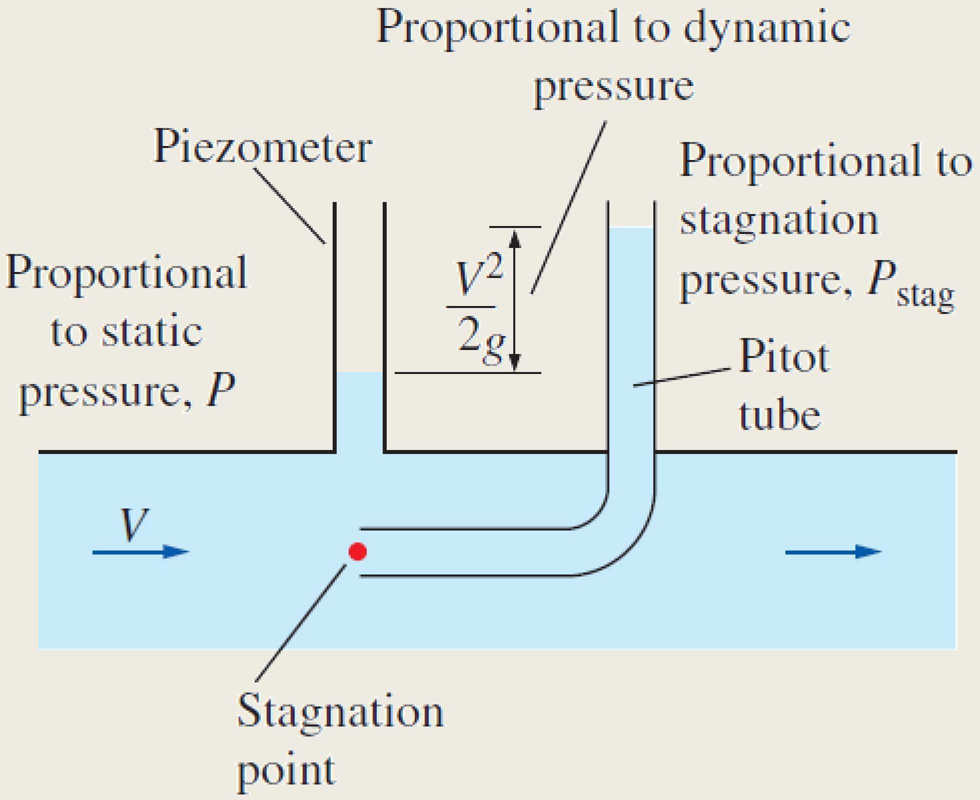
$$V \approx \left[2\frac{(p_0 - p_s)}{\rho}\right]^{1/2}$$



Pitot tube installed on a plane

Piezometer tube

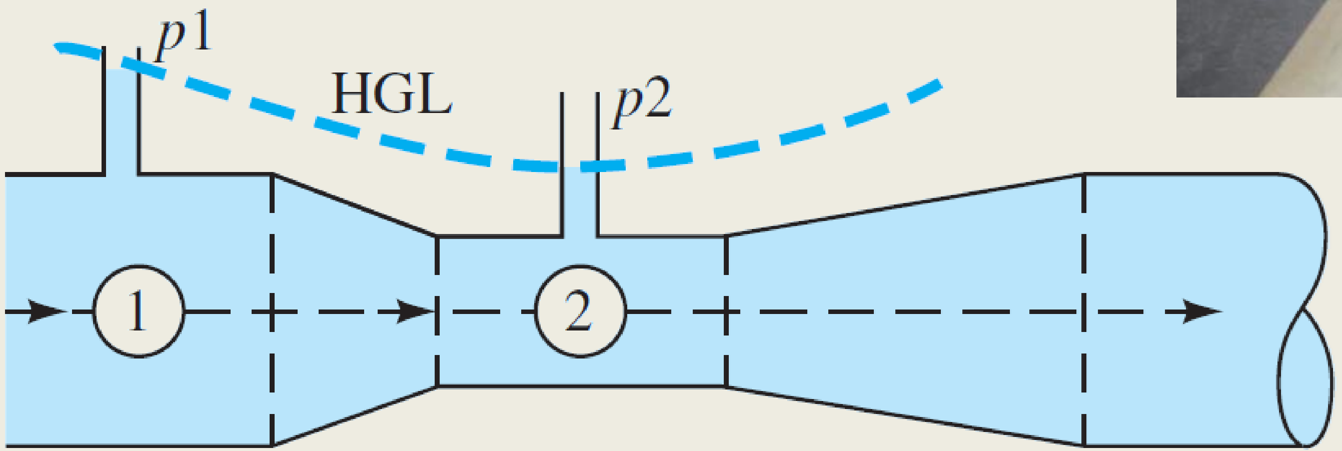
- For liquids at pressure greater than atmospheric



$$V = \sqrt{\frac{2(P_{stag} - P)}{\rho}}$$

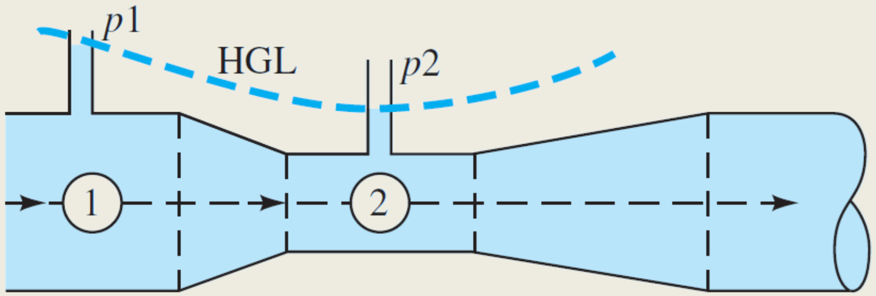
Example: Venturi tube

This device is called venturi tube which is used for measuring the flow rate through the pipe. Find an expression for the mass flow in the tube as a function of the pressure change.



https://www.drurylandetheatre.com/venturi-flow-meter/

Bernoulli equation between points 1 and 2 ($z_1 = z_2$):



$$\frac{p_1}{\rho} + \frac{1}{2}V_1^2 + gz_1 = \frac{p_2}{\rho} + \frac{1}{2}V_2^2 + gz_2$$

$$V_2^2 - V_1^2 = \frac{2\,\Delta p}{\rho} \qquad \Delta p = p_1 - p_2$$

Continuity equation for incompressible flow: $A_1V_1 = A_2V_2$

$$V_1 = \beta^2 V_2 \qquad \beta = \frac{D_2}{D_1} \Rightarrow V_2 = \left[\frac{2\,\Delta p}{\rho(1-\beta^4)}\right]^{1/2} \Rightarrow \dot{m} = \rho A_2 V_2 = A_2\left(\frac{2\rho\,\Delta p}{1-\beta^4}\right)^{1/2}$$

Venturi experiment