MECHANICS OF FLUIDS

Lecture 9 – Differential Analysis

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https://web.stanford.edu/group/fpc/cgi-bin/fpcwiki/Research/SupersonicJet

Note

- All the art-work contents of this lecture are obtained from the following sources, unless otherwise stated:
  - *Fluid Mechanics, 8th edition, Frank M. White, McGraw-Hill, 2016.*
  - *Fluid Mechanics: Fundamental and Applications, 3rd edition, Yunus A. Cengel, John M. Cimbala, McGraw-Hill, 2014.*

Motivation

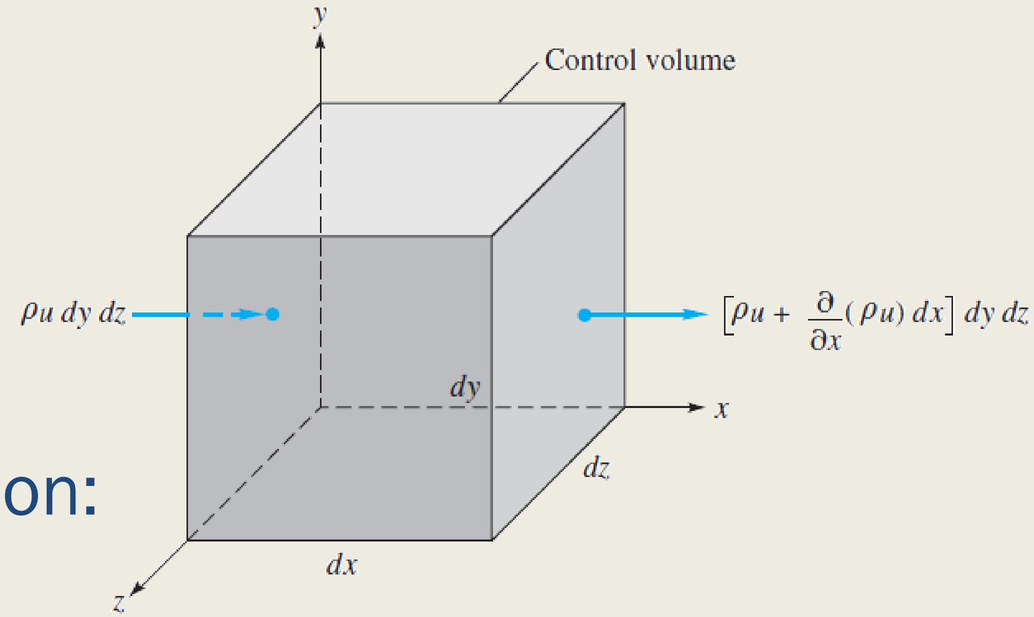
- We need more information about flow pattern in the process
  - *Reactive flows, multiphase flows*

- The flow is so complex that the integral formulation may incur the analysis
  - *Jet flows, compressible combustible flow, turbulent flows*

- We perform differential analysis to obtain parameters that can be used for integral formulation.
  - *Drag force, heat transfer coefficient, friction loss*

Mass conservation (Continuity)

- Consider a differential control volume with six faces

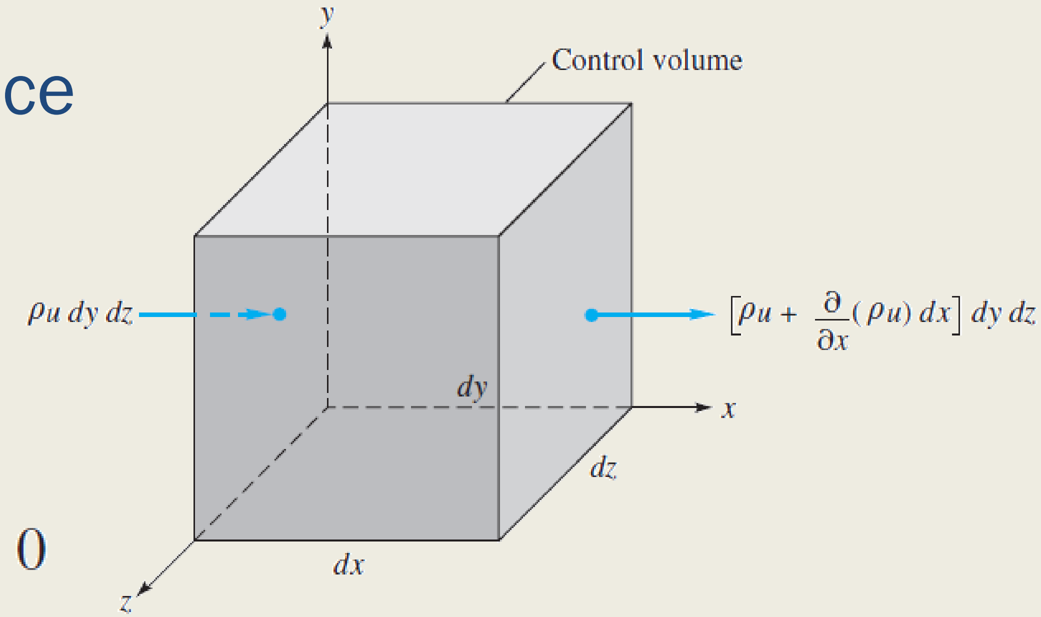


- From previous lectures, we have the following equation for mass conservation:

$$\underbrace{\int_{CV} \frac{\partial \rho}{\partial t} \, d\mathcal{V}}_{\text{Rate of mass change in the CV}} + \underbrace{\int_{CS} \rho(\mathbf{V} \cdot \mathbf{n}) \, dA}_{\text{Rate of mass out of the CV through faces – Rate of mass into the CV through face}} = 0$$

Mass conservation (Continuity)

- Since the CV is differential, each surface is very small, we can consider uniform flow at each face:



$$\underbrace{\int_{CV} \frac{\partial \rho}{\partial t}\, d\mathcal{V}}_{} + \sum_i (\rho_i A_i V_i)_{out} - \sum_i (\rho_i A_i V_i)_{in} = 0$$

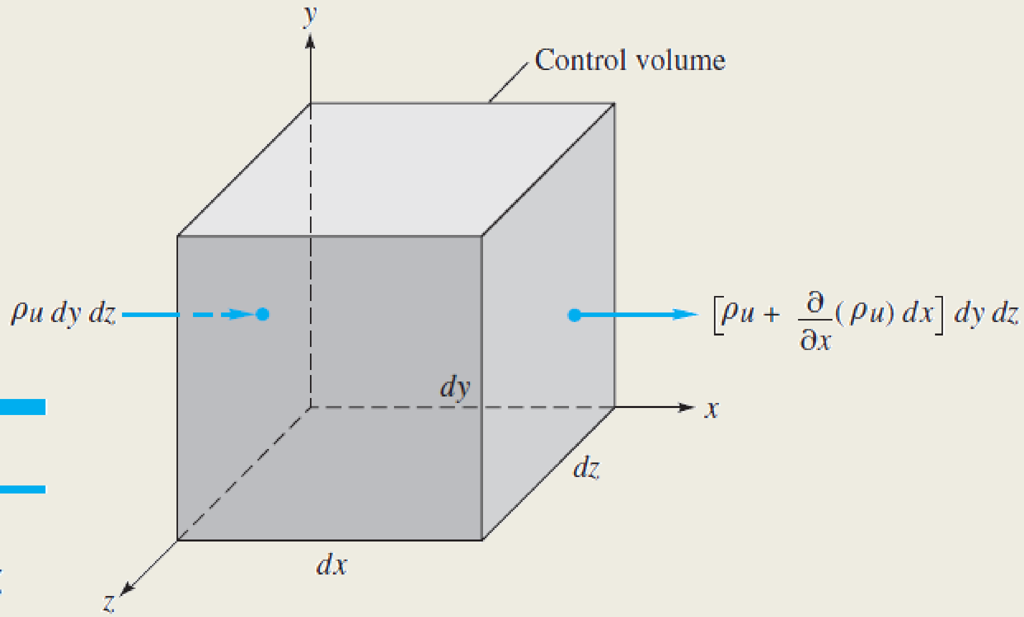
$dV \rightarrow 0$

$$\int_{CV} \frac{\partial \rho}{\partial t}\, d\mathcal{V} \approx \frac{\partial \rho}{\partial t}\, dx\, dy\, dz$$

The rate of fluid mass change **in the CV** = Sum of all mass flow rates **into** the CV — Sum of all mass flow rates **out of** the CV

Mass conservation (Continuity)

- Flow terms can be estimates as:



Face	Inlet mass flow	Outlet mass flow
x	$\rho u \, dy \, dz$	$\left[\rho u + \frac{\partial}{\partial x}(\rho u)\, dx\right] dy \, dz$
y	$\rho v \, dx \, dz$	$\left[\rho v + \frac{\partial}{\partial y}(\rho v)\, dy\right] dx \, dz$
z	$\rho w \, dx \, dy$	$\left[\rho w + \frac{\partial}{\partial z}(\rho w)\, dz\right] dx \, dy$

Recall : $F(x+dx) = F(x) + \frac{d}{dx}\big(F(x)\big)dx + \frac{1}{2!}\frac{d^2}{dx^2}\big(F(x)\big)dx^2 + \cdots$

Mass conservation (Continuity)

Rate of mass change in the CV

Sum of mass flow rate out of the CV – Sum of mass flow rate into the CV

$$\overbrace{\frac{\partial \rho}{\partial t}\,dx\,dy\,dz}^{\text{Rate of mass change in the CV}} + \overbrace{\frac{\partial}{\partial x}(\rho u)\,dx\,dy\,dz + \frac{\partial}{\partial y}(\rho \upsilon)\,dx\,dy\,dz + \frac{\partial}{\partial z}(\rho w)\,dx\,dy\,dz}^{\text{Sum of mass flow rate out of the CV – Sum of mass flow rate into the CV}} = 0$$

$\div\, dx\,dy\,dz$

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho \upsilon) + \frac{\partial}{\partial z}(\rho w) = 0$$

Continuity equation for Cartesian coordinates

Mass conservation (Continuity)

- Introducing the divergence operator:

$$\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) \equiv \nabla \cdot (\rho \mathbf{V})$$

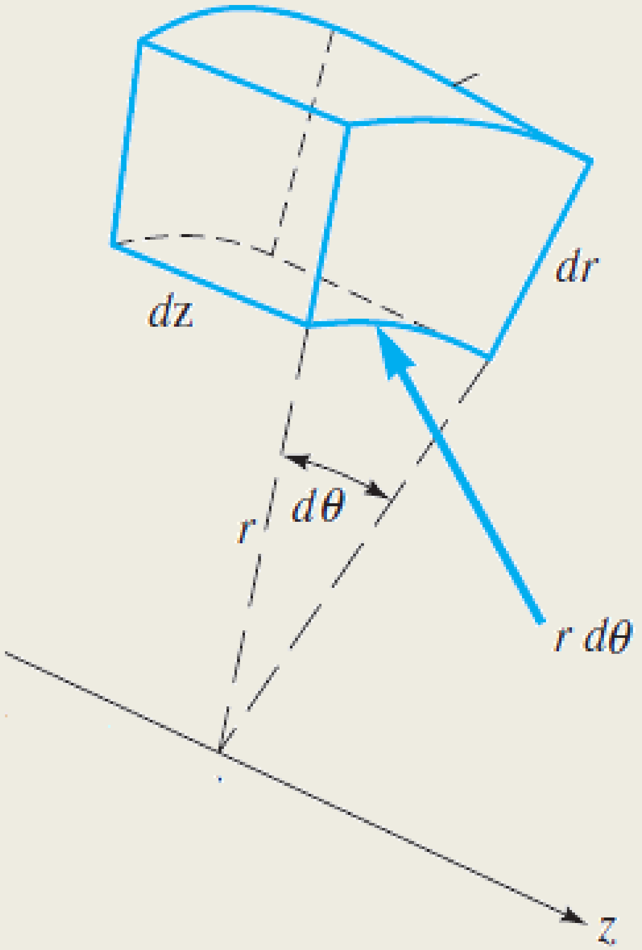
- The general continuity equation becomes:

$$\boxed{\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V}) = 0}$$

Mass conservation (Continuity)

- For cylindrical coordinates:

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(r\rho v_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(\rho v_\theta) + \frac{\partial}{\partial z}(\rho v_z) = 0$$



Continuity equation

- Steady compressible flow: $\nabla \cdot (\rho \mathbf{V}) = 0$

Cartesian: $$\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0$$

Cylindrical: $$\frac{1}{r}\frac{\partial}{\partial r}(r\rho v_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(\rho v_\theta) + \frac{\partial}{\partial z}(\rho v_z) = 0$$

- Incompressible flow: $\nabla \cdot \mathbf{V} = 0$

Cartesian: $$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

Cylindrical: $$\frac{1}{r}\frac{\partial}{\partial r}(r v_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(v_\theta) + \frac{\partial}{\partial z}(v_z) = 0$$

Linear Momentum Analysis

Material Derivation of velocity

- Consider that the velocity filed in a fluid varies with space and time:

$$\mathbf{V}(\mathbf{r}, t) = \mathbf{i}u(x, y, z, t) + \mathbf{j}\upsilon(x, y, z, t) + \mathbf{k}w(x, y, z, t)$$

- The acceleration of the fluid then becomes:

$$\mathbf{a} = \frac{d\mathbf{V}}{dt} = \mathbf{i}\frac{du}{dt} + \mathbf{j}\frac{d\upsilon}{dt} + \mathbf{k}\frac{dw}{dt}$$

- For x-component we have:

$$\frac{du(x, y, z, t)}{dt} = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}\underbrace{\frac{dx}{dt}}_{u} + \frac{\partial u}{\partial y}\underbrace{\frac{dy}{dt}}_{\upsilon} + \frac{\partial u}{\partial z}\underbrace{\frac{dz}{dt}}_{w}$$

Material derivative of velocity

$$a_x = \frac{du}{dt} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \upsilon\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = \frac{\partial u}{\partial t} + (\mathbf{V}\cdot\nabla)u$$

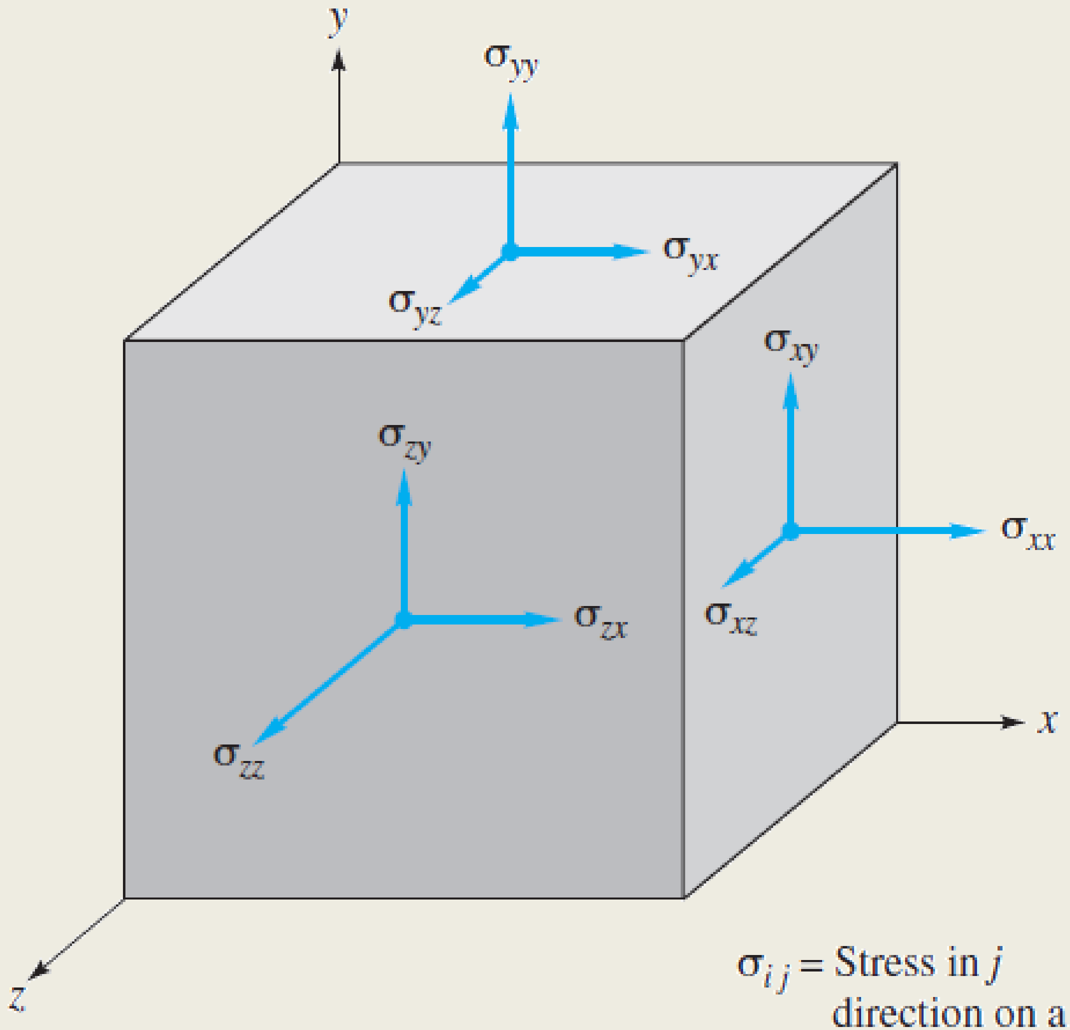
$$a_y = \frac{d\upsilon}{dt} = \frac{\partial \upsilon}{\partial t} + u\frac{\partial \upsilon}{\partial x} + \upsilon\frac{\partial \upsilon}{\partial y} + w\frac{\partial \upsilon}{\partial z} = \frac{\partial \upsilon}{\partial t} + (\mathbf{V}\cdot\nabla)\upsilon$$

$$a_z = \frac{dw}{dt} = \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + \upsilon\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = \frac{\partial w}{\partial t} + (\mathbf{V}\cdot\nabla)w$$

- and in vector form

$$\boxed{\mathbf{a} = \frac{d\mathbf{V}}{dt} = \underbrace{\frac{\partial \mathbf{V}}{\partial t}}_{\text{Local}} + \underbrace{\left(u\frac{\partial \mathbf{V}}{\partial x} + \upsilon\frac{\partial \mathbf{V}}{\partial y} + w\frac{\partial \mathbf{V}}{\partial z}\right)}_{\text{Convective}} = \frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V}\cdot\boldsymbol{\nabla})\mathbf{V}}$$

Stress tensor



σ_{ij} = Stress in j direction on a face normal to i axis

$$\sigma_{ij} = \begin{vmatrix} -p + \tau_{xx} & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & -p + \tau_{yy} & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & -p + \tau_{zz} \end{vmatrix}$$

Stress tensor

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Linear momentum equation

- Recall the force balance equation for the differential fluid element in the Cartesian coordinates (lecture 2):

$$\boxed{\sum \mathbf{f} = \mathbf{f}_{\text{press}} + \mathbf{f}_{\text{grav}} + \mathbf{f}_{\text{visc}} = -\nabla p + \rho \mathbf{g} + \mathbf{f}_{\text{visc}} = \rho \mathbf{a}}$$

- Or equivalently, we can say for a differential control volume:

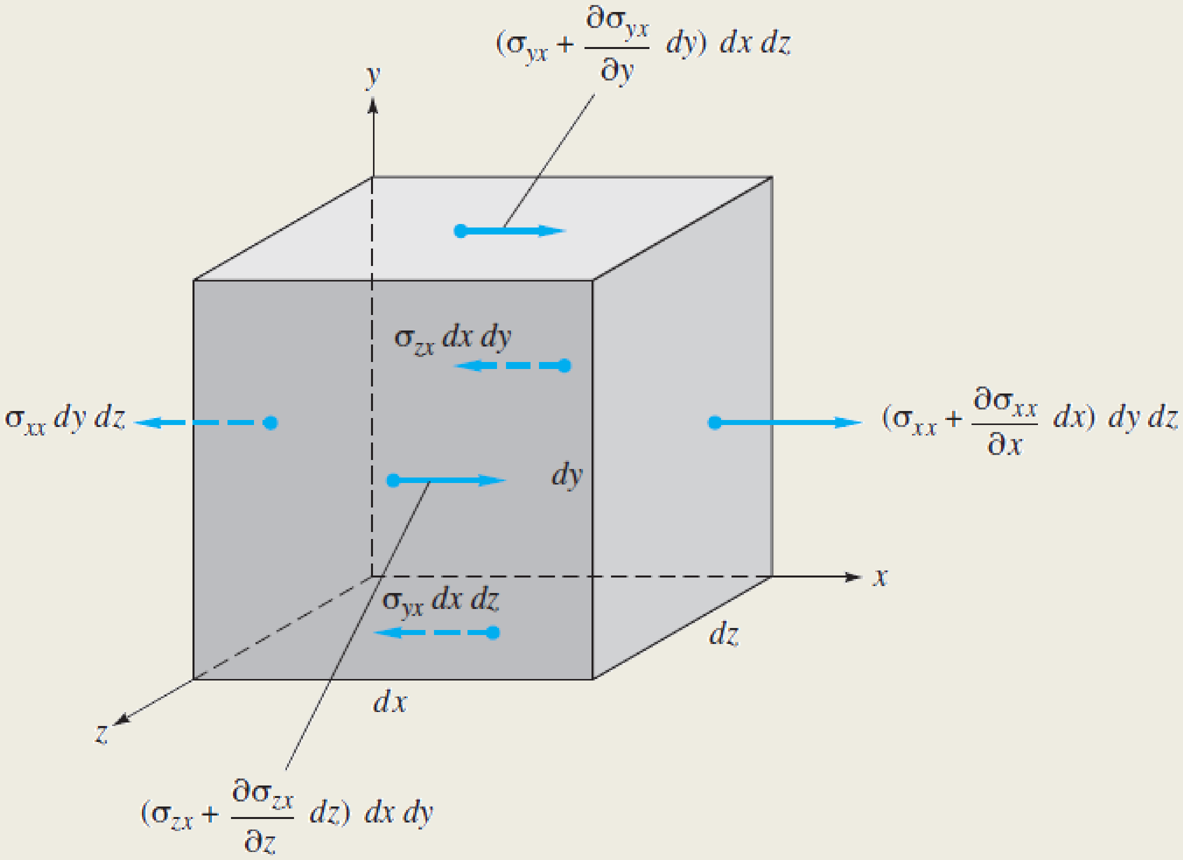
$$\sum \mathbf{F} = \rho \frac{d\mathbf{V}}{dt} \, dx \, dy \, dz$$

Sum of all surface forces and body forces (act on volume)

- Weight
- Pressure and stress

Linear momentum equation

- The net surface force acting on the CV in x-direction:



Note:
Recall our direction convention,
Stress is propagated from greater x to lesser x

Linear momentum equation

$$dF_{x,\text{surf}} = \left[\frac{\partial}{\partial x}(\sigma_{xx}) + \frac{\partial}{\partial y}(\sigma_{yx}) + \frac{\partial}{\partial z}(\sigma_{zx})\right] dx\,dy\,dz$$

$$\Downarrow$$

$$\frac{dF_x}{d\mathcal{V}} = -\frac{\partial p}{\partial x} + \frac{\partial}{\partial x}(\tau_{xx}) + \frac{\partial}{\partial y}(\tau_{yx}) + \frac{\partial}{\partial z}(\tau_{zx})$$

- Similarly for y and z directions:

$$\frac{dF_y}{d\mathcal{V}} = -\frac{\partial p}{\partial y} + \frac{\partial}{\partial x}(\tau_{xy}) + \frac{\partial}{\partial y}(\tau_{yy}) + \frac{\partial}{\partial z}(\tau_{zy})$$

$$\frac{dF_z}{d\mathcal{V}} = -\frac{\partial p}{\partial z} + \frac{\partial}{\partial x}(\tau_{xz}) + \frac{\partial}{\partial y}(\tau_{yz}) + \frac{\partial}{\partial z}(\tau_{zz})$$

Linear momentum equation

- And finally sum of surface forces in three directions:

$$\left(\frac{d\mathbf{F}}{d\mathcal{V}}\right)_{\text{surf}} = -\boldsymbol{\nabla} p + \left(\frac{d\mathbf{F}}{d\mathcal{V}}\right)_{\text{viscous}}$$

$$\left(\frac{d\mathbf{F}}{d\mathcal{V}}\right)_{\text{viscous}} = \boldsymbol{\nabla}\cdot\boldsymbol{\tau}_{ij} = \mathbf{i}\left(\frac{\partial\tau_{xx}}{\partial x}+\frac{\partial\tau_{yx}}{\partial y}+\frac{\partial\tau_{zx}}{\partial z}\right) + \mathbf{j}\left(\frac{\partial\tau_{xy}}{\partial x}+\frac{\partial\tau_{yy}}{\partial y}+\frac{\partial\tau_{zy}}{\partial z}\right) + \mathbf{k}\left(\frac{\partial\tau_{xz}}{\partial x}+\frac{\partial\tau_{yz}}{\partial y}+\frac{\partial\tau_{zz}}{\partial z}\right)$$

$$\tau_{ij} = \begin{bmatrix} \tau_{xx} & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & \tau_{yy} & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & \tau_{zz} \end{bmatrix}$$

Viscous stress tensor

$$\tau_{ij} = \begin{bmatrix} \tau_{xx} & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & \tau_{yy} & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & \tau_{zz} \end{bmatrix}$$

Components of viscous shear stress

Linear momentum equation

- And the body force on the control volume is:

$$d\mathbf{F}_{\text{grav}} = \rho\mathbf{g}\, dx\, dy\, dz$$

- Then the general force balance equation becomes:

$$\sum \mathbf{F} = \rho \frac{d\mathbf{V}}{dt} dx\, dy\, dz$$

Substitution from previous relations and dividing by (dx.dy.dz)

$$\rho\mathbf{g} - \nabla p + \nabla \cdot \boldsymbol{\tau}_{ij} = \rho\left(\frac{\partial \boldsymbol{V}}{\partial t} + u\frac{\partial \boldsymbol{V}}{\partial t} + v\frac{\partial \boldsymbol{V}}{\partial t} + w\frac{\partial \boldsymbol{V}}{\partial t}\right)$$

Linear momentum equation

- In component form for cartesian coordinates:

$$\rho g_x - \frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} = \rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \upsilon\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right)$$

$$\rho g_y - \frac{\partial p}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} = \rho\left(\frac{\partial \upsilon}{\partial t} + u\frac{\partial \upsilon}{\partial x} + \upsilon\frac{\partial \upsilon}{\partial y} + w\frac{\partial \upsilon}{\partial z}\right)$$

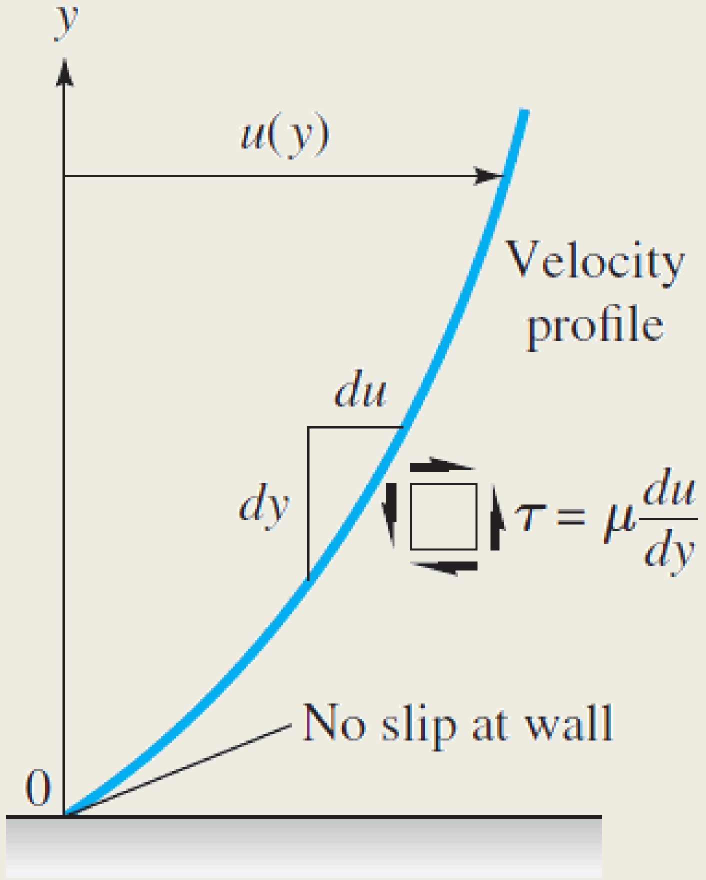
$$\rho g_z - \frac{\partial p}{\partial z} + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z} = \rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + \upsilon\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right)$$

Navier-Stokes equations

- In parallel plate flow, we showed that the τ_{xy} is related to the strain rate for a Newtonian fluid.

- Where the stress acts from **greater y** to the **lesser y.**

$$\tau = \mu \frac{du}{dy}$$



Navier-Stokes equations

- In a 3D flow, for incompressible, Newtonian fluid we have:

$$\underbrace{\tau_{ij}}_{\text{Viscous stress}} = 2\underbrace{\mu}_{\text{Dynamic viscosity}}\underbrace{\varepsilon_{ij}}_{\text{Strain rate}}$$

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} \qquad \varepsilon_{yy} = \frac{\partial v}{\partial y} \qquad \varepsilon_{zz} = \frac{\partial w}{\partial z}$$

$$\varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \qquad \varepsilon_{zx} = \frac{1}{2}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) \qquad \varepsilon_{yz} = \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)$$

Navier-Stokes equations

- Viscous stress tensor for a general viscous flow and for an **incompressible, Newtonian** fluid:

$$\tau_{ij} = \begin{pmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{pmatrix} = \begin{pmatrix} 2\mu \dfrac{\partial u}{\partial x} & \mu\left(\dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x}\right) & \mu\left(\dfrac{\partial u}{\partial z} + \dfrac{\partial w}{\partial x}\right) \\ \mu\left(\dfrac{\partial v}{\partial x} + \dfrac{\partial u}{\partial y}\right) & 2\mu \dfrac{\partial v}{\partial y} & \mu\left(\dfrac{\partial v}{\partial z} + \dfrac{\partial w}{\partial y}\right) \\ \mu\left(\dfrac{\partial w}{\partial x} + \dfrac{\partial u}{\partial z}\right) & \mu\left(\dfrac{\partial w}{\partial y} + \dfrac{\partial v}{\partial z}\right) & 2\mu \dfrac{\partial w}{\partial z} \end{pmatrix}$$

Navier-Stokes equations

- Consider x-component of momentum equation:

$$\rho g_x - \frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} = \rho \left(\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + \upsilon \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} \right)$$

- The left-hand side becomes:

$$-\frac{\partial P}{\partial x} + \rho g_x + 2\mu \frac{\partial^2 u}{\partial x^2} + \mu \frac{\partial}{\partial y} \left(\underline{\frac{\partial v}{\partial x}} + \frac{\partial u}{\partial y} \right) + \mu \frac{\partial}{\partial z} \left(\underline{\frac{\partial w}{\partial x}} + \frac{\partial u}{\partial z} \right)$$

Navier-Stokes equations

- By changing the order of derivatives in the underlined terms:

$$-\frac{\partial P}{\partial x}+\rho g_x+\mu\left[\frac{\partial^2 u}{\partial x^2}+\underline{\frac{\partial}{\partial x}\frac{\partial u}{\partial x}}+\underline{\frac{\partial}{\partial x}\frac{\partial v}{\partial y}}+\frac{\partial^2 u}{\partial y^2}+\underline{\frac{\partial}{\partial x}\frac{\partial w}{\partial z}}+\frac{\partial^2 u}{\partial z^2}\right]$$

$$-\frac{\partial P}{\partial x}+\rho g_x+\mu\left[\frac{\partial}{\partial x}\underbrace{\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)}_{\text{Is zero, continuity equation}}+\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right]$$

Navier-Stokes equations

- In general, the momentum equations in all directions take the form, which are known as **Navier-Stokes** equations.

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right)=-\frac{\partial P}{\partial x}+\rho g_x+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)$$

$$\rho\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\right)=-\frac{\partial P}{\partial y}+\rho g_y+\mu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right)$$

$$\rho\left(\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right)=-\frac{\partial P}{\partial z}+\rho g_z+\mu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right)$$

Navier-Stokes equations in Cartesian coordinates

Navier-Stokes equations

- In cylindrical coordinates:

$$\tau_{ij} = \begin{pmatrix} \tau_{rr} & \tau_{r\theta} & \tau_{rz} \\ \tau_{\theta r} & \tau_{\theta\theta} & \tau_{\theta z} \\ \tau_{zr} & \tau_{z\theta} & \tau_{zz} \end{pmatrix}$$

$$= \begin{pmatrix} 2\mu \dfrac{\partial u_r}{\partial r} & \mu\left[r\dfrac{\partial}{\partial r}\left(\dfrac{u_\theta}{r}\right) + \dfrac{1}{r}\dfrac{\partial u_r}{\partial \theta}\right] & \mu\left(\dfrac{\partial u_r}{\partial z} + \dfrac{\partial u_z}{\partial r}\right) \\ \mu\left[r\dfrac{\partial}{\partial r}\left(\dfrac{u_\theta}{r}\right) + \dfrac{1}{r}\dfrac{\partial u_r}{\partial \theta}\right] & 2\mu\left(\dfrac{1}{r}\dfrac{\partial u_\theta}{\partial \theta} + \dfrac{u_r}{r}\right) & \mu\left(\dfrac{\partial u_\theta}{\partial z} + \dfrac{1}{r}\dfrac{\partial u_z}{\partial \theta}\right) \\ \mu\left(\dfrac{\partial u_r}{\partial z} + \dfrac{\partial u_z}{\partial r}\right) & \mu\left(\dfrac{\partial u_\theta}{\partial z} + \dfrac{1}{r}\dfrac{\partial u_z}{\partial \theta}\right) & 2\mu \dfrac{\partial u_z}{\partial z} \end{pmatrix}$$

Navier-Stokes equations

$$\frac{1}{r}\frac{\partial(ru_r)}{\partial r} + \frac{1}{r}\frac{\partial(u_\theta)}{\partial \theta} + \frac{\partial(u_z)}{\partial z} = 0$$

Continuity

$$\rho\left(\frac{\partial u_r}{\partial t} + u_r\frac{\partial u_r}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta^2}{r} + u_z\frac{\partial u_r}{\partial z}\right)$$
$$= -\frac{\partial P}{\partial r} + \rho g_r + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_r}{\partial r}\right) - \frac{u_r}{r^2} + \frac{1}{r^2}\frac{\partial^2 u_r}{\partial \theta^2} - \frac{2}{r^2}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial^2 u_r}{\partial z^2}\right]$$

r-component

$$\rho\left(\frac{\partial u_\theta}{\partial t} + u_r\frac{\partial u_\theta}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r u_\theta}{r} + u_z\frac{\partial u_\theta}{\partial z}\right)$$
$$= -\frac{1}{r}\frac{\partial P}{\partial \theta} + \rho g_\theta + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_\theta}{\partial r}\right) - \frac{u_\theta}{r^2} + \frac{1}{r^2}\frac{\partial^2 u_\theta}{\partial \theta^2} + \frac{2}{r^2}\frac{\partial u_r}{\partial \theta} + \frac{\partial^2 u_\theta}{\partial z^2}\right]$$

θ-component

Navier-Stokes equations

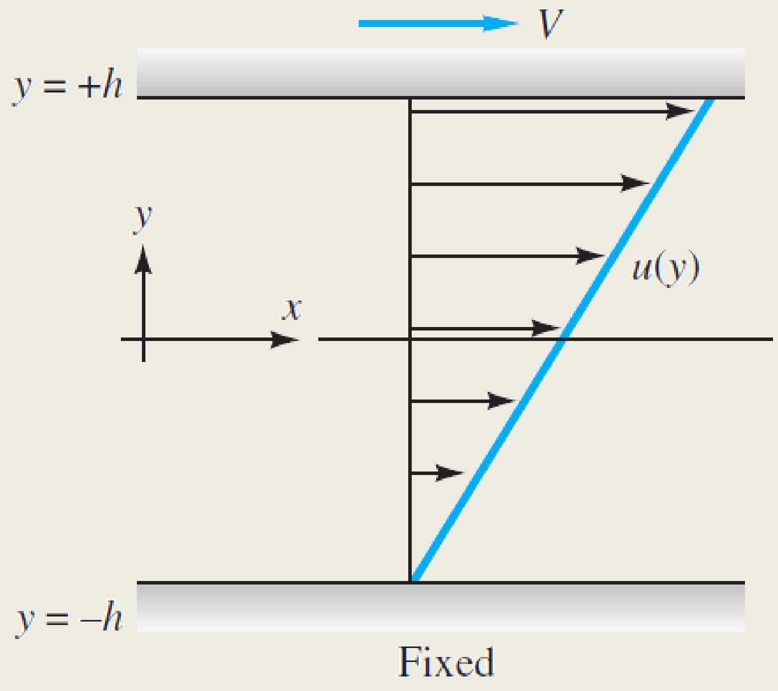
$$\rho\left(\frac{\partial u_z}{\partial t} + u_r\frac{\partial u_z}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_z}{\partial \theta} + u_z\frac{\partial u_z}{\partial z}\right)$$

z-component

$$= -\frac{\partial P}{\partial z} + \rho g_z + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u_z}{\partial \theta^2} + \frac{\partial^2 u_z}{\partial z^2}\right]$$

Flow between parallel plates (Couette flow)

Consider 2D incompressible flow between two parallel plates with the distance 2h. We assume plates are too wide and long and hence, $v = 0$ and $w = 0$. Find the velocity distribution between these two plates at fully developed condition.



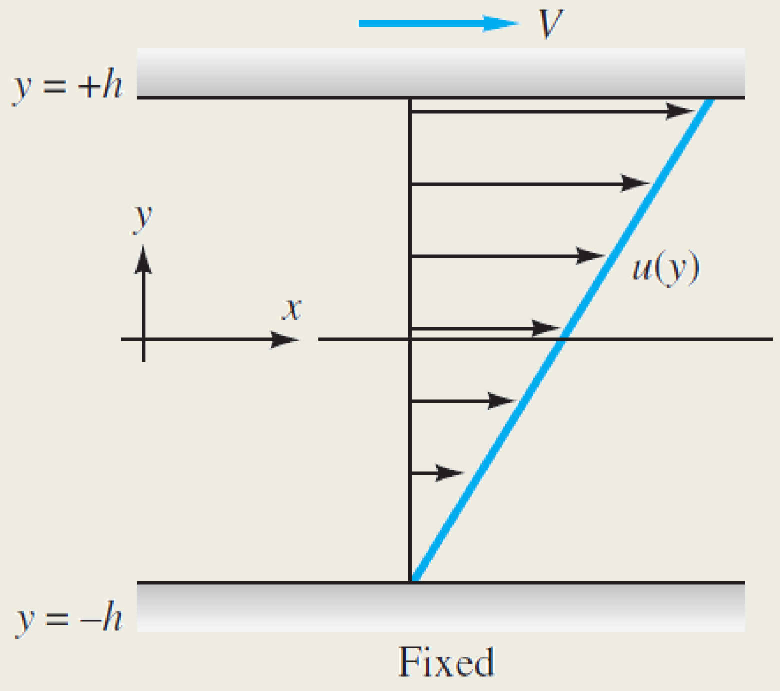
Flow between parallel plates (Couette flow)

- Since the flow is two dimensional, velocity components are function of *x* and *y*, only *u* and *v* components present.
- From continuity equation for incompressible flow:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 = \frac{\partial u}{\partial x} + 0 + 0$$

- This shows that *u* is a function of *y* only.

$$u = u(y) \text{ only}$$



Flow between parallel plates (Couette flow)

The **fully developed** laminar conditions implies that, the flow should be steady, thus there is not change with respect to time.

The x-component momentum equation for Newtonian fluid (2D version) at steady conditions:

$$\rho\left(\underbrace{\cancel{u\frac{\partial u}{\partial x}}}_{0} + \underbrace{\cancel{v\frac{\partial u}{\partial y}}}_{0}\right) = \underbrace{\cancel{-\frac{\partial p}{\partial x}}}_{0} + \underbrace{\cancel{\rho g_x}}_{0} + \mu\left(\underbrace{\cancel{\frac{\partial^2 u}{\partial x^2}}}_{0} + \frac{\partial^2 u}{\partial y^2}\right)$$

$$\frac{d^2 u}{dy^2} = 0 \quad \Rightarrow \quad u = C_1 y + C_2$$

Two boundary conditions at walls (no-slip condition):

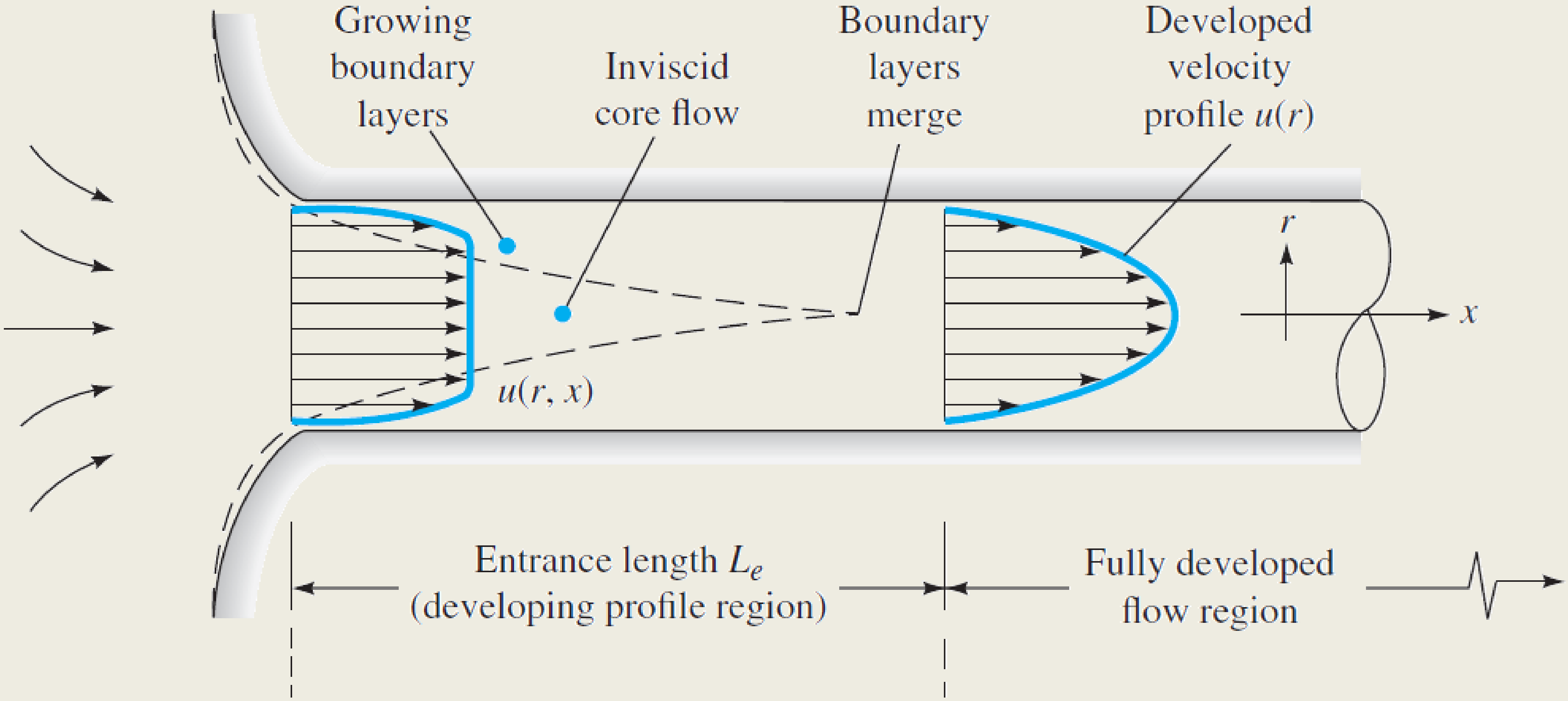
$(1)\ y = -h,\ u = 0$

$(2)\ y = h,\ u = V$

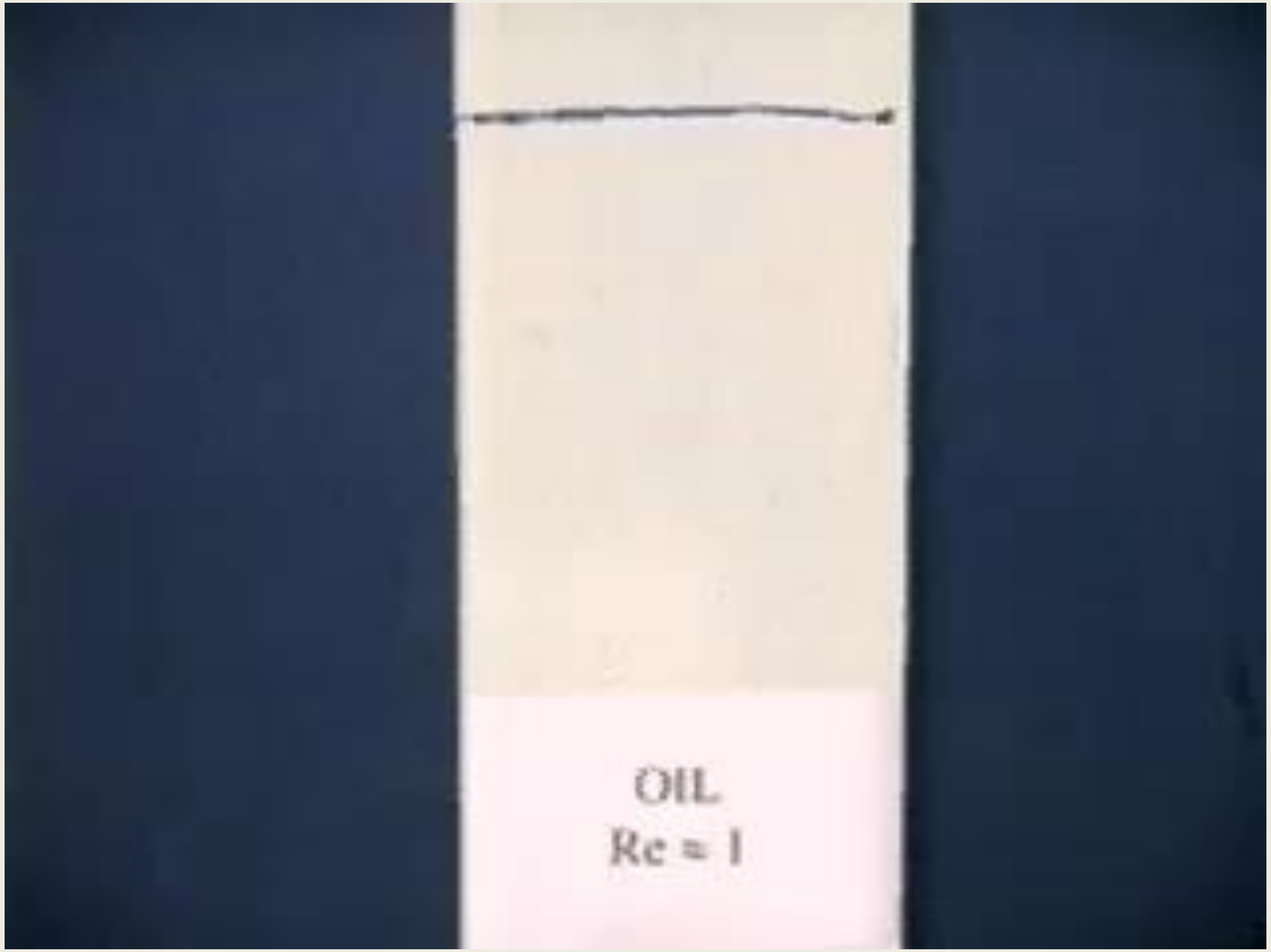
Applying boundary conditions, we get:

$$C_1 = \frac{V}{2h} \quad \text{and} \quad C_2 = \frac{V}{2} \quad \Rightarrow \quad \boxed{u = \frac{V}{2h}y + \frac{V}{2}}$$

Fully developed Laminar flow in pipe

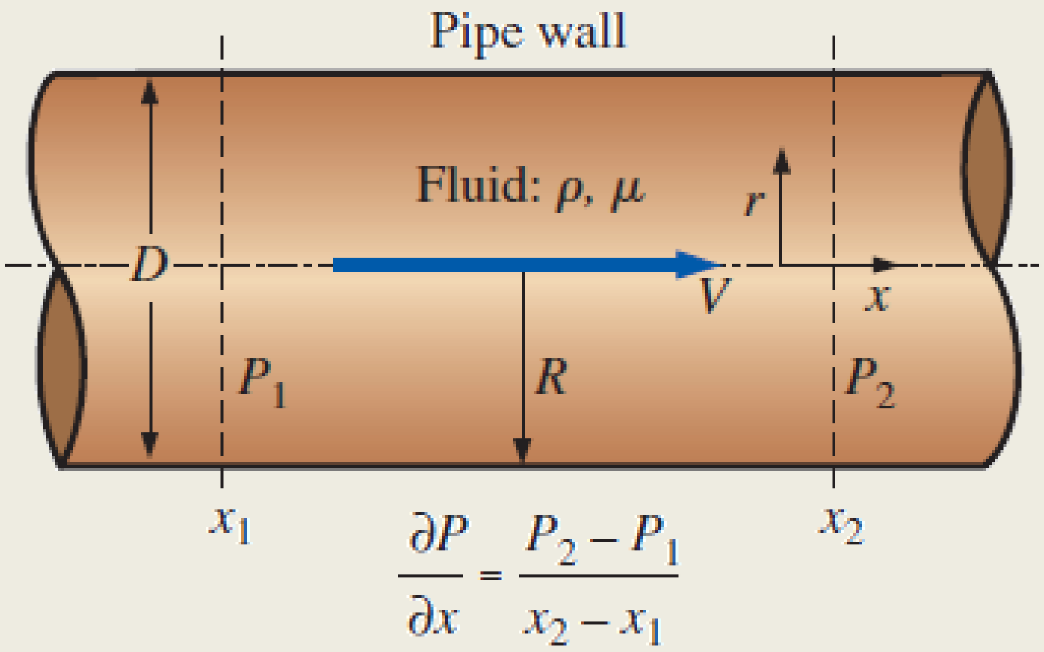


Fully developed Laminar flow in pipe



Example (pipe flow)

Consider a **laminar flow** in a pipe wherein a constant pressure gradient is applied in **x-direction**, that cause the flow. Derive and expression for the **steady** velocity field inside the pipe at **fully developed** condition.



Example (pipe flow)

- **Assumptions:**
  1. *The pipe is infinitely long in the x-direction.*
  2. *The flow is steady (all partial time derivatives are zero).*
  3. *This is a parallel flow (the r-component of velocity, ur, is zero).*
  4. *The fluid is incompressible and Newtonian with constant properties, and the flow is laminar.*
  5. *A constant pressure gradient is applied in the x-direction.*
  6. *The velocity field is axisymmetric with no swirl, implying that* u_θ *is zero.*
  7. *We ignore the effects of gravity.*

Example (pipe flow)

Continuity equation for compressible flow:

$$\underbrace{\cancel{\frac{1}{r}\frac{\partial(r u_r)}{\partial r}}}_{\text{assumption 3}} + \underbrace{\cancel{\frac{1}{r}\frac{\partial(u_\theta)}{\partial \theta}}}_{\text{assumption 6}} + \frac{\partial u}{\partial x} = 0 \quad \rightarrow \quad \frac{\partial u}{\partial x} = 0$$

Since the flow is not a function of time (steady condition) and θ (assumption 6), we can conclude that:

$$u = u(r) \text{ only}$$

Example (pipe flow)

$$\rho\left(\underbrace{\cancel{\frac{\partial u}{\partial t}}}_{\text{assumption 2}} + \underbrace{u_r \cancel{\frac{\partial u}{\partial r}}}_{\text{assumption 3}} + \underbrace{\cancel{\frac{u_\theta}{r}\frac{\partial u}{\partial \theta}}}_{\text{assumption 6}} + \underbrace{u\cancel{\frac{\partial u}{\partial x}}}_{\text{continuity}}\right)$$

$$= -\frac{\partial P}{\partial x} + \underbrace{\cancel{\rho g_x}}_{\text{assumption 7}} + \mu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \underbrace{\cancel{\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}}}_{\text{assumption 6}} + \underbrace{\cancel{\frac{\partial^2 u}{\partial x^2}}}_{\text{continuity}}\right)$$

$$\boxed{\frac{1}{r}\frac{d}{dr}\left(r\frac{du}{dr}\right) = \frac{1}{\mu}\frac{\partial P}{\partial x}}$$

Example (pipe flow)

Integrating the previous ODE for two times gives:

$$r\frac{du}{dr} = \frac{r^2}{2\mu}\frac{dP}{dx} + C_1 \quad \Rightarrow \quad \boxed{u = \frac{r^2}{4\mu}\frac{dP}{dx} + C_1 \ln r + C_2}$$

Boundary conditions:

(1) r = R → u = 0 (no-slip conditions)

(2) r = 0 → du/dr = 0 (symmetry in the profile or the value of u should be finite at center)

Example (pipe flow)

- Boundary condition #2:

$$C_1 = 0$$

- Boundary condition #1:

$$u = \frac{R^2}{4\mu}\frac{dP}{dx} + 0 + C_2 = 0 \quad \rightarrow \quad C_2 = -\frac{R^2}{4\mu}\frac{dP}{dx}$$

- Substitution into the equation and rearranging give:

$$u = \frac{1}{4\mu}\frac{dP}{dx}(r^2 - R^2)$$ ⟵ Verify it!

Example (pipe flow)

- Maximum velocity at r = 0:

$$u_{\max} = -\frac{R^2}{4\mu}\frac{dP}{dx}$$

- Average volume flow rate:

$$\dot{V} = \int_{\theta=0}^{2\pi}\int_{r=0}^{R} ur\,dr\,d\theta =$$

$$\frac{2\pi}{4\mu}\frac{dP}{dx}\int_{r=0}^{R}(r^2 - R^2)r\,dr = -\frac{\pi R^4}{8\mu}\frac{dP}{dx}$$

Example (pipe flow)

Average axial velocity:

$$V = \frac{\dot{V}}{A} = \frac{(-\pi R^4/8\mu)\,(dP/dx)}{\pi R^2} = -\frac{R^2}{8\mu}\frac{dP}{dx}$$

Pressure drop for a segment with the length *L*:

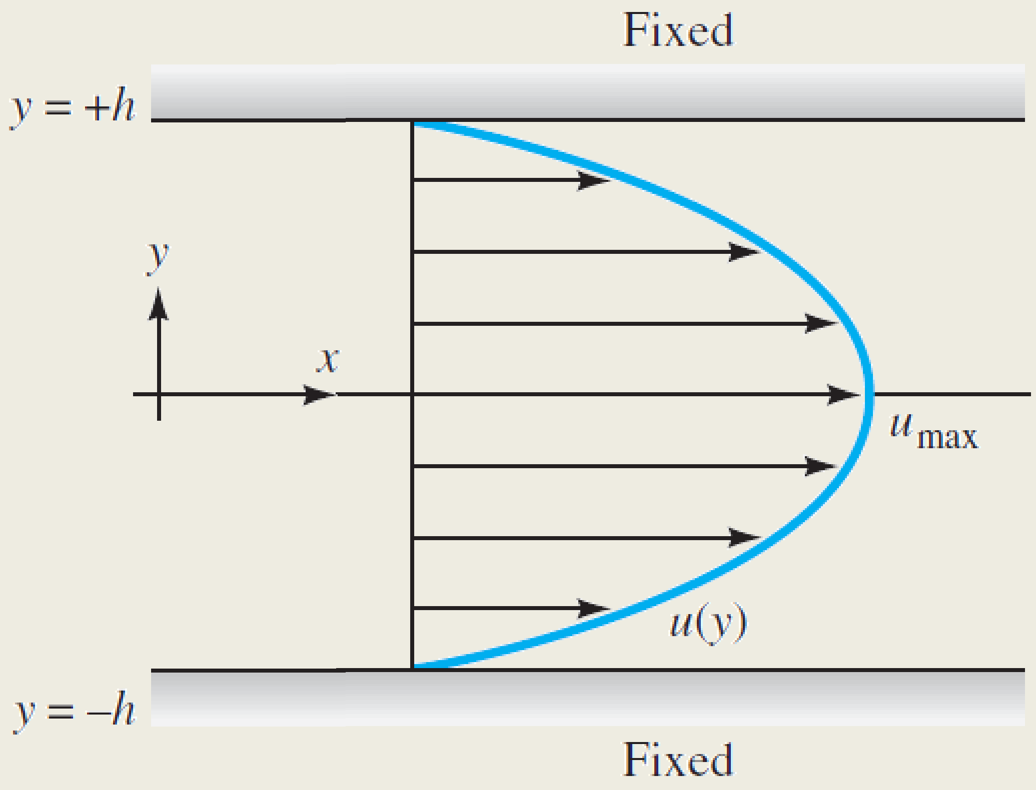
$$V = \frac{D^2}{32\,\mu}\frac{\Delta P}{L} \qquad \Delta P = \left(\frac{L}{D}\right)\left(\frac{32\,\mu\,V}{D}\right)$$

$\div\ \frac{1}{2}\rho V^2$

$$\boxed{\frac{\Delta P}{\frac{1}{2}\rho V^2} = \left(\frac{L}{D}\right)\left(\frac{64\,\mu}{\rho V D}\right) = \left(\frac{L}{D}\right)\left(\frac{64}{Re}\right)}$$

Flow between two fixed plates due to pressure gradient

Consider a fluid between two infinite parallel plates which move due to pressure gradient along x-axis. Find the velocity distribution of the fluid at fully developed condition.



Flow between two fixed plates due to pressure gradient

- Since $v = 0$ and $w = 0$, we have the following from continuity equation:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 = \frac{\partial u}{\partial x} + 0 + 0$$

- The x-component of momentum equation:

$$\rho\left(\underbrace{\cancel{u\frac{\partial u}{\partial x}}}_{0} + \underbrace{\cancel{v\frac{\partial u}{\partial y}}}_{0}\right) = -\frac{\partial p}{\partial x} + \underbrace{\cancel{\rho g_x}}_{0} + \mu\left(\underbrace{\cancel{\frac{\partial^2 u}{\partial x^2}}}_{0} + \frac{\partial^2 u}{\partial y^2}\right) \Rightarrow \boxed{\mu\frac{d^2 u}{dy^2} = \frac{\partial p}{\partial x}}$$

Flow between two fixed plates due to pressure gradient

This implies that:

$$\mu \frac{d^2u}{dy^2} = \frac{dp}{dx} = \text{const} < 0$$

Double integration from above equation:

$$u = \frac{1}{\mu}\frac{dp}{dx}\frac{y^2}{2} + C_1 y + C_2$$

Two boundary conditions at walls (no-slip condition):

$$y = -h,\ u = 0$$
$$y = h,\ \ u = 0$$

$$\Rightarrow$$

$$C_1 = 0$$

$$C_2 = -\frac{dp}{dx}\frac{h^2}{2\mu}$$

Flow between two fixed plates due to pressure gradient

The velocity profile:

$$u = -\frac{dp}{dx}\frac{h^2}{2\mu}\left(1 - \frac{y^2}{h^2}\right)$$

And the maximum velocity occurs at y = 0:

$$u_{\text{max}} = -\frac{dp}{dx}\frac{h^2}{2\mu}$$

Average velocity across the channel (the depth is b):

$$V_{\text{av}} = \frac{1}{A}\int u\,dA = \frac{1}{b(2h)}\int_{-h}^{+h} u_{\text{max}}\left(1 - \frac{y^2}{h^2}\right) b\,dy = \frac{2}{3}\,u_{\text{max}}$$

Flow between two fixed plates due to pressure gradient

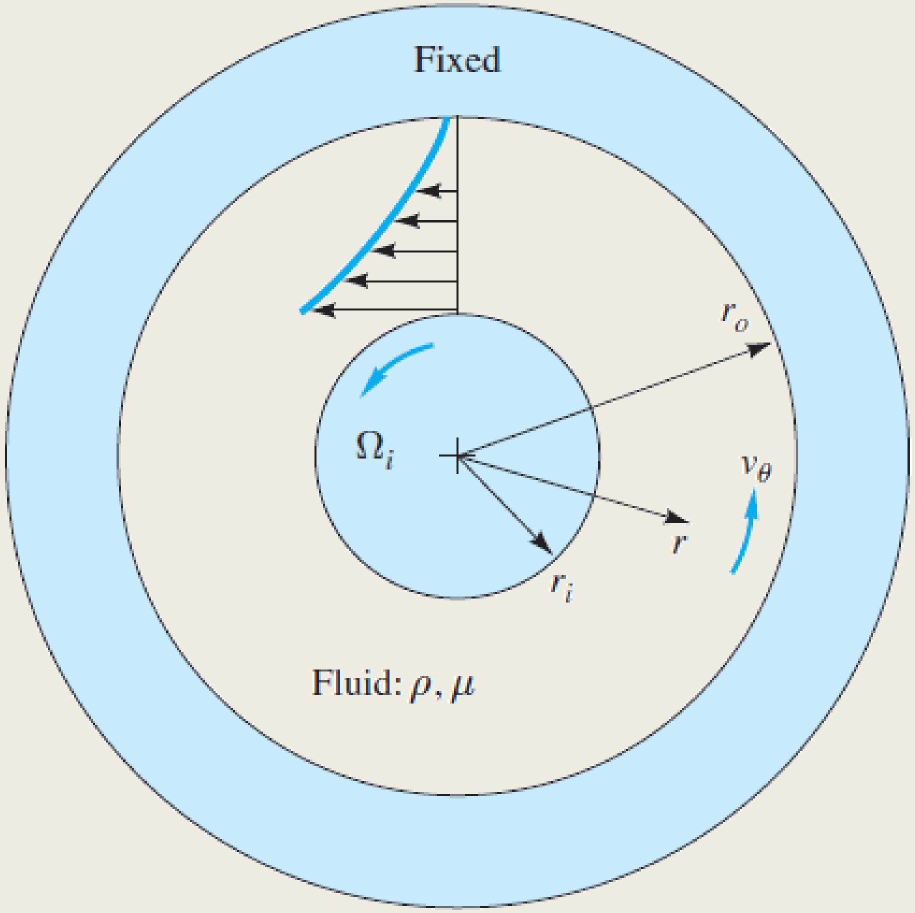
Shear stress at walls

$$\tau_w = \tau_{xy\,\text{wall}} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial \upsilon}{\partial x}\right)\bigg|_{y\,=\,\pm h} = \mu\frac{\partial}{\partial y}\left[\left(-\frac{dp}{dx}\right)\left(\frac{h^2}{2\mu}\right)\left(1 - \frac{y^2}{h^2}\right)\right]\bigg|_{y=\,\pm h}$$

$$= \pm\frac{dp}{dx}h = \mp\frac{2\mu u_{\max}}{h}$$

Viscous flow in a rotary viscometer

Consider a fluid of constant density and viscosity between two concentric cylinders. There is no axial motion or end effect. Let the inner cylinder rotate at angular velocity Ω_i . Let the outer cylinder be fixed. There is circular symmetry, so the velocity does not vary with θ and varies only with *r*.



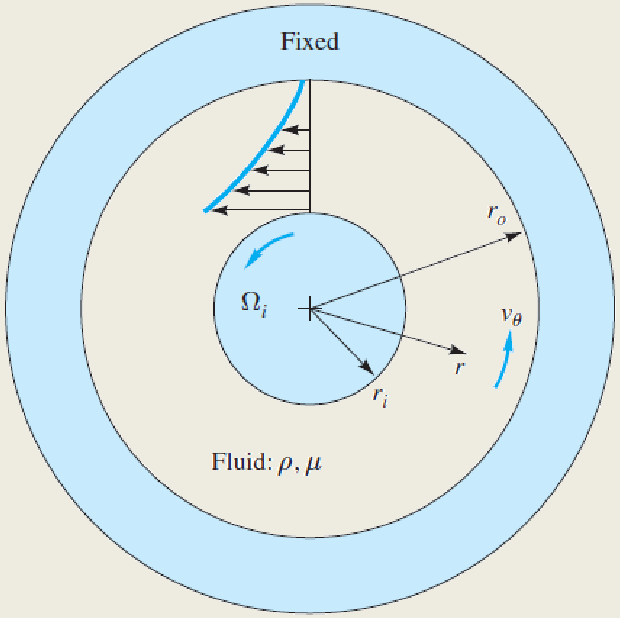
Viscous flow in a rotary viscometer

$$\frac{1}{r}\frac{\partial(ru_r)}{\partial r} + \cancelto{0}{\frac{1}{r}\frac{\partial(u_\theta)}{\partial\theta}} + \cancelto{0}{\frac{\partial(u_z)}{\partial z}} = 0 \;\Rightarrow\; 0 = \frac{1}{r}\frac{d}{dr}(r\upsilon_r)$$

$$\Rightarrow\; r\upsilon_r = \text{const}$$

$\upsilon_r = 0$ Walls are impermeable

$$\rho\left(\cancelto{0}{\frac{\partial u_\theta}{\partial t}} + \cancelto{0}{u_r\frac{\partial u_\theta}{\partial r}} + \cancelto{0}{\frac{u_\theta}{r}\frac{\partial u_\theta}{\partial\theta}} + \cancelto{0}{\frac{u_r u_\theta}{r}} + \cancelto{0}{u_z\frac{\partial u_\theta}{\partial z}}\right)$$
$$= \cancelto{0}{-\frac{1}{r}\frac{\partial P}{\partial\theta}} + \cancelto{0}{\rho g_\theta} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_\theta}{\partial r}\right) - \frac{u_\theta}{r^2} + \cancelto{0}{\frac{1}{r^2}\frac{\partial^2 u_\theta}{\partial\theta^2}} + \cancelto{0}{\frac{2}{r^2}\frac{\partial u_r}{\partial\theta}} + \cancelto{0}{\frac{\partial^2 u_\theta}{\partial z^2}}\right]$$



Viscous flow in a rotary viscometer

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dv_\theta}{dr}\right) = \frac{v_\theta}{r^2}$$

Cauchy-Euler Equation

$m^2 - 1 = 0 \Rightarrow m = \pm 1$

$$v_\theta = C_1 r + \frac{C_2}{r}$$

Boundary conditions

Outer, at $r = r_o$: $\quad v_\theta = 0 = C_1 r_o + \frac{C_2}{r_o}$

Inner, at $r = r_i$: $\quad v_\theta = \Omega_i r_i = C_1 r_i + \frac{C_2}{r_i}$

$$\boxed{v_\theta = \Omega_i r_i \frac{r_o/r - r/r_o}{r_o/r_i - r_i/r_o}}$$

Cauchy-Euler

$$x^2\frac{d^2y}{dx^2} + ax\frac{dy}{dx} + by = 0$$

$$m^2 + (a-1)m + b = 0.$$